Physics 330 – Problem Set # 8

(due Thursday, December 4)

1. You may know the Optical Theorem from your Quantum Mechanics course: In a scattering process, the imaginary part of the amplitude for forward scattering is proportional to the total cross section. The physical interpretation of this result is that, when two particles interact, the total probability that something happens is 1. If there is scattering, then the probability of scattering must be compensated by decreasing the probability that the two particles go past one another. This is accomplished by the destructive interference of the forward scattering amplitude with the 1 term in the S-matrix. In this problem, you will work out the proof more carefully.

   (a) The S matrix is time-evolution; thus, it is a unitary matrix. Write $S = 1 + iT$. Convert the expression of unitarity

   $$S^\dagger S = 1 \tag{1}$$

   into a relation for T. Take the matrix element of this equation between 2-particle states. Insert

   $$\langle p_1' p_2' | \, iT \, | p_1 p_2 \rangle = i\mathcal{M} (2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2) \tag{2}$$

   and write this as an equation for $\mathcal{M}$.

   (b) In the limit $(p_1', p_2') \to (p_1, p_2)$, interpret the square of $\mathcal{M}$ in terms of the total cross section. This will give the relation

   $$\text{Im}\ \mathcal{M} = 2E_{CM} p_{CM}\, \sigma_{tot}\ . \tag{3}$$

   which is a very concrete statement of the optical theorem.

2. In ϕ^4 theory, the theory of real-valued scalar fields with mass m and

   $$H_{int} = \int d^3x\, \frac{\lambda}{4!} \phi^4\ , \tag{4}$$

   compute the $\phi\phi$ scattering amplitude to 1-loop order. You can do this rather explicitly using the ideas from our analysis of 1-loop diagrams in QED. You will see that all UV divergences appearing at this order can be subsumed into two directly measurable observables, the mass of the ϕ boson m and the value of the cross section at the threshold $s = 4m^2$. (There are no IR divergences in this theory.)

   (a) Compute the scattering amplitude for ϕ^4 theory to order λ, and compute the total cross section to order λ^2. (By this point in the course, this should be almost trivial.)

(b) The order λ correction to the ϕ propagator is given by the diagram

$$-iM^2 =$$  (5)

Show that this diagram is a constant, independent of momenta. (It is UV divergent, but it can be regularized.) Show that the mass of the ϕ is then given by

$$m^2 = m_0^2 + M^2 \tag{6}$$

where m_0^2 is the mass given in the Hamiltonian. Then the diagram (5)—and its divergent value—is not observable. Show also that $Z = 1$ to this order.

(c) Draw the remaining diagrams that contribute to the $\phi\phi \to \phi\phi$ scattering amplitude at order λ^2. (There are 3 of these diagrams.) The diagrams are UV divergent, but they can be made finite with Pauli-Villars regularization

$$\frac{i}{p^2 - m^2} \to \frac{i}{p^2 - m^2} - \frac{i}{p^2 - \Lambda^2} \ . \tag{7}$$

Compute the diagrams and write the scattering amplitude to order λ^2. You can write the amplitude in terms of the function

$$f(Z) = \int_0^1 dx \ \log(1 - x(1-x)Z) \ . \tag{8}$$

It is straightforward to evaluate this integral, but the result is not very enlightening.

(d) Let $\mathcal{M}_0$ be the value of the scattering amplitude at threshold $s = 4m^2, t = 0, u = 0$. Rewrite the scattering amplitude as a sum of $\mathcal{M}_0$ and a function of s, t, u that vanishes at threshold. Show that this function has no divergences and so is a well-defined prediction of ϕ^4 theory.

(e) Verify the optical theorem for this process to order λ^2 from the explicit expression for the scattering amplitude in (d). It is helpful to realize, from the expression for the loop diagrams, that the variable s should be evaluated at $s + i\epsilon$.