Physics 330 – Problem Set # 7

(due Thursday, November 20)

1. The Higgs boson has a coupling to fermions (quarks and leptons) given by the interaction Hamiltonian
$$\Delta H = \int d^3x \, \frac{y_f}{\sqrt{2}} h \, \overline{\Psi}_f \Psi_f \ , \qquad (1)$$
where $h(x)$ is the Higgs boson field and Ψ_f is the Dirac field for the fermion of type f. The coefficient y_f, called the Yukawa coupling" is given by
$$y_f = \sqrt{2} \frac{m_f}{v} \ , \qquad (2)$$
where m_f is the mass of the fermion and $v = 246$ GeV. The Higgs boson has a nonzero mass, measured to be $m_h = 125$ GeV.

   (a) Draw the Feynman diagram for the 1-loop vertex correction to the electromagnetic current due to exchange of a virtual Higgs boson.

   (b) Compute this diagram formally as an integral over Feynman parameters x, y, z. Show that it falls into the standard form for a correction to the electromagnetic current,
$$\overline{u}(p')[\gamma^\mu \delta F_1(q^2) + i\frac{\sigma^{\mu\nu} q_\nu}{2m_f} F_2(q^2)]u(p) \qquad (3)$$

   (c) Write $F_2(0)$ as an integral over 1 Feynman parameter. Evaluate $F_2(0)$ in the limit in which the Higgs boson mass is negligible with respect to m_f.

   (d) Evaluate the leading term in $F_2(0)$ in the limit $m_h \gg m_f$.

   (e) Unfortunately, there are no leptons heavier than the τ lepton, with mass $m_\tau = 1.777$ GeV. Evaluate the contribution to the anomalous magnetic moment of the τ lepton due to Higgs boson exchange.

2. In class, we studied the singularity of the formula for emission of a photon from a scattered electron in the limit where the photon mass becomes very small (as in reality). This is called the "soft" singularity. However, when the initial and final electron have energies greater than m_e, this formula has a second logarithmic singularity, called the "collinear" singularity. In this problem, you can work out the origin of both singularities.

   (a) Draw the Feynman diagrams for 1-photon emission in electron scattering from a fixed charge. Label the diagram with p as the final electron 4-momentum and k as the 4-momentum of the emitted photon. Write $P = p + k$.

(b) In class, we showed that, both diagrams are singular when $k \to 0$. This is called the "soft" singularity. However, there is another singularity, called the "collinear" singularity. In the diagram in which the photon is emitted from the outgoing electron line, there is a denominator P^2. Show that P^2 becomes small as the 3-momenta p and k become parallel, even when the E_p and E_k are large. Show that the diagram is truly infinite in this kinematics if we ignore the electron mass. (We come close to the singularity when $E_p \gg m_e$.) Show that there is a similar collinear singularity in the second diagram, when $\vec{k}$ is parallel to the incoming electron direction. From here on, set $m_e = 0$. This means that most of the radiation is emitted either near the direction of the final electron ("final-state radiation") or near the direction of the initial electron ("initial-state radiation").

These two singularities are in different regions of phase space, so, to a first approximation, they can be treated incoherently. In the following, we will analyze the singularity in the direction of the outgoing electron. Then we will need only the first Feynman diagram.

(c) Choosing the direction of P as the $\hat{3}$ axis, and the $(\hat{1}, \hat{3})$ plane as the event plane, write the three momenta as

$$\begin{aligned} p &= (E_p, p_\perp, 0, E_p - p_\perp^2/2E_p) \\ k &= (E_k, -p_\perp, 0, E_k - p_\perp^2/2E_k) \\ P &= (E, 0, 0, E - p_\perp^2/2E_p - p_\perp^2/2E_k) \end{aligned} \tag{4}$$

Note that I have put the final particles on shell and then evaluated P using momentum conservation. Since energy is conserved, write

$$E_k = zE \qquad E_p = (1-z)E \qquad \text{where } \; 0 < z < 1 \tag{5}$$

Evaluate P^2; this is of the order of $p_\perp^2$.

(d) Assume that the initial (virtual) electron is left-handed, with spinor

$$u(P) = \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} . \tag{6}$$

The final electron spin will then also be left-handed, but rotated through an angle $\theta_p = p_\perp / E_p$. For the photon polarizations, use the helicity eigenvectors

$$\begin{aligned} \epsilon_R^\mu &= (0, 1, -i, 0)^\mu / \sqrt{2} \\ \epsilon_L^\mu &= (0, 1, i, 0)^\mu / \sqrt{2} \end{aligned} \tag{7}$$

rotated through the angle $\theta_k = -p_\perp / E_k$. Write out the final spinor and the two needed polarization vectors. In the following, it will suffice to work with these to order $p_\perp^1$.

(e) Write the fermion propagator as

$$\frac{i \not{P}}{P^2} = i \sum_s u^s(P)\overline{u}^s(P)\ \frac{1}{P^2} \tag{8}$$

and insert P^2 from part (b). Near the singularity, the spinors are well-approximated by their on-shell values. The index $s = L, R$, but only L will be relevant if the initial electron is left-handed. Show that the diagram then falls apart into two pieces, one proportional to the leading order scattering amplitude and one with an emission vertex

$$(-ie)\overline{u}(p)\gamma \cdot \epsilon^*(k)u(P) \tag{9}$$

(f) Evaluate the emission vertex to order $p_\perp^1$ for each of the two photon polarizations.

(g) Square and sum the results, include the factor $(1/P^2)^2$ and the leading order matrix element for the scattering amplitude, and integrate over the final phase space. You can convert the integral d^3p to an integral d^3P using $d^3k d^3p = d^3k d^3P$, since $P = p + k$. The factor $2p$ in the denominator of the phase space integral can be written $2p = 2(1 - z)P$.

After all of these manipulations, show that the cross section for scattering with one photon in the final-state radiation region takes the form:

$$d\sigma = d\sigma_0 \cdot \frac{\alpha}{2\pi} \int \frac{dp_\perp^2}{p_\perp^2} \int dz\ \frac{1 + (1 - z)^2}{z}. \tag{10}$$

The expression multiplying $d\sigma_0$ gives the probability of final-state radiation. Each integral gives a logarithmic singularity. The $p_\perp^2$ integral represents the collinear singularity; the z integral represents the soft singularity. This is actually a universal formula for the dominant contribution to the emission of photons in any reaction with relativistic initial or final electrons.