Physics 330 – Problem Set # 6

(due Thursday, November 13)

1. In class, I showed a way to simplify the expression for 2-body relativistic phase space. In this problem, you can simplify the expression for 3-body relativistic phase space.

   For this problem, let 1, 2, 3 denote the three final-state particles, with masses M_1, M_2, M_3 and (on-shell) 4-momenta p_1, p_2, p_3. Let $Q = p_1+p_2+p_3$ be the total 4-momentum. Work in the center of mass (CM) frame, where $Q = (E_{CM}, 0, 0, 0)$. Then

$$\int d\Pi_3 = \int \frac{d^3p_1 d^3p_2 d^3p_3}{(2\pi)^9\, 2E_1\, 2E_2\, 2E_3}(2\pi)^4\delta^{(4)}(Q-(p_1+p_2+p_3)) \ . \tag{1}$$

   (a) Show that the three final-state particles lie in a plane, the *event plane.* Let θ_{12} be the angle between $\vec{p}_1$ and $\vec{p}_2$, and define θ_{13} and θ_{23} similarly.

   (b) Define

$$x_1 = \frac{2p_1\cdot Q}{Q^2} \qquad x_2 = \frac{2p_2\cdot Q}{Q^2} \qquad x_3 = \frac{2p_3\cdot Q}{Q^2} \ , \tag{2}$$

   Show that $x_1 + x_2 + x_3 = 2$. Write expressions for the CM energies E_i and the CM momenta $|\vec{p}_i|$ ($i = 1, 2, 3$) in terms of the x_i and the particle masses.

   (c) Show that the invariant mass of the system 1+2 is given by

$$(p_1+p_2)^2 = m_{12}^2 = (1-x_3)Q^2 + M_3^2 \tag{3}$$

   (d) Find an expression for $\cos\theta_{12}$ in terms of the x_i. Show that, up to the orientation of the event plane, the final state is completely fixed by specifying the three x_i.

   (e) The formula for 3-body phase space contains 9 integrals and 4 delta functions. Three of the integrals are integrals over the orientation of the event plane. Integrate over these. Remove three of the delta functions by integrating over d^3p_3. In the remaining, energy-conserving, delta function, the energy E_3 is now a function of $|\vec{p}_1 + \vec{p}_2|$ and therefore is a function of $\cos\theta_{12}$ Eliminate this delta function by integrating over $\cos\theta_{12}$.

   (f) The two remaining integrals can be written as integrals over x_1 and x_2. Show that everything else cancels, so that

$$\int d\Pi_3 = \frac{Q^2}{128\pi^3}\int dx_1 dx_2 \tag{4}$$

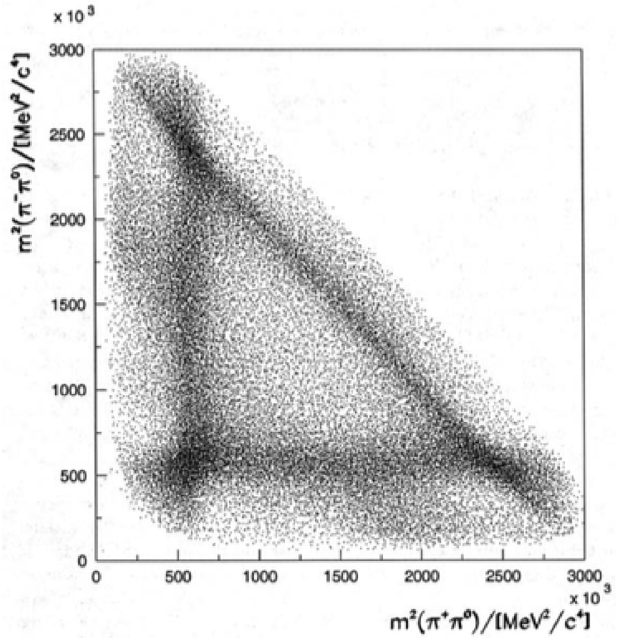
   (g) Show that (4) can alternatively be written as

$$\int d\Pi_3 = \frac{1}{128\pi^3 Q^2}\int dm_{23}^2 dm_{13}^2 \ , \tag{5}$$

with

$$m_{12}^2 + m_{13}^2 + m_{23}^2 = Q^2 + M_1^2 + M_2^2 + M_3^2 \tag{6}$$

The formula (5) is amazing. It implies that, if the scattering matrix element is a constant, the 2-particle masses will have a flat distribution when plotted in the (m_{13}^2, m_{23}^2) plane. Any peak in this plane observed in experimental data must then be the result of actual particle interactions. This plot is called the “Dalitz plot”. Here is the Dalitz plot for proton-antiproton annihilation to 3 pi mesons at threshold:



(7)

The heavy bands denote the ρ meson, which appears as a resonance at 775 MeV ($m_\rho^2 = 600 \times 10^3$ MeV^2) in each 2-pi meson channel.

(h) One issue with the Dalitz plot is that it is not so simple to determine where its boundaries are. Start with the case $M_1 = M_2 = M_3 = 0$. The region of the x_1, x_2 plane allowed for 3-body phase space is that in which all three x_i satisfy $0 < x_i < 1$. Draw this region. Notice that, on its boundaries, all three momenta lie on a line, with two momenta parallel and the other recoiling against them. Draw this region in the m_{13}^2, m_{23}^2 plane. You will notice that this region is very similar to that filled in the figure, because the pi meson mass is much less than the proton mass.

(i) Use this idea to find the boundaries in the case $M_1 = M_2 = 0$ but $M_3 > 0$.

2. We now have enough technology to compute the lifetime of the muon. The muon decays through the weak interactions, and probably you know nothing about these, but you can do this computation if I give you the relevant Feynman rules. You will also need to know the formula for the decay rate Γ of an unstable particle of mass M. This is almost identical to the formula for the cross section,

$$\Gamma = \frac{1}{2M} \int d\Pi \; |\mathcal{M}|^2 \tag{8}$$

with the final state treated in the same way as for the cross section and the flux factor replaced by the factor $(2M)$ from the relativistic normalization of the initial state.

(a) The muon is a Dirac fermion that decays via $\mu^- \to e^- \nu_\mu \overline{\nu}_e$, where the ν_i are different species of neutrinos. The decay process can be described by the perturbation

$$\int \Delta H = \int d^3x \left\{ \frac{G_F}{\sqrt{2}} \, \overline{\Psi}_{\nu_\mu} \gamma^\alpha (1-\gamma^5) \Psi_\mu \; \overline{\Psi}_e \gamma_\alpha (1-\gamma^5) \Psi_{\nu_e} \right\} . \tag{9}$$

where $G_F = 1.166 \times 10^{-5}$ GeV^{-2} (determined from the rate of beta decay of unstable nuclei). In this equation, the Ψ_a are four Dirac fermion fields: Ψ_μ is the field that annihilates the muon, $\overline{\Psi}_e$ is the field that creates the electron, $\overline{\Psi}_{\nu_\mu}$ is the field that creates the muon-type neutrino, and Ψ_{ν_e} is the field that annihilates an electron-type neutrino or creates an electron-type antineutrino. Since the muon mass is much larger than all of the other masses, you may treat the electron and the two neutrinos as massless. Write the Feynman rule for this vertex.

(b) Write out the decay matrix element $\mathcal{M}$. Square it, and simplify the result using the formulae for traces of gamma matrices, averaging over the muon spin and summing over the electron and neutrino spins.

(c) Using the formulae from problem 1, write the decay rate as an integral over

$$x_e = \frac{E_e}{m_\mu} \quad \text{and} \quad x_{\overline{\nu}} = \frac{E_{\overline{\nu}_e}}{m_\mu} \tag{10}$$

(d) The emitted neutrinos are essentially invisible, so it is only possible to measure the energy distribution of the decay electron. To predict this distribution, carry out the integral over $x_{\overline{\nu}}$ and find

$$\frac{d\Gamma_\mu}{dx_e} \tag{11}$$

(e) Integrate over x_e to find the decay rate of the muon. Convert the lifetime to seconds and compare to the measured value $\tau_\mu = 2.2 \times 10^{-6}$ s.

(f) Using an angular momentum argument, show that, when $x_e = 1$, so that the electron recoils directly against the two neutrinos, the decay rate vanishes when the electron direction is parallel to the muon spin direction. This feature is observed experimentally.