Physics 330 – Problem Set # 5

(due Thursday, October 30)

1. Using the formalism that we have developed, compute the differential cross section $d\sigma/d\cos\theta$ for the Coulomb scattering of an electron from a very heavy charged particle at rest.

   (a) Let the initial 4-momentum of the electron be $(E, \vec{p})$ and the final 4-momentum be $(E', \vec{p}')$. The momentum transfered to the heavy charged particle is $\vec{q} = \vec{p} - \vec{p}'$. Show that, if the mass of the heavy particle M is much greater than $|\vec{q}|$, then the heavy particle gains negligible energy, and $E = E'$. Work out the expression for 2-body phase space in this limit in the lab frame in which the heavy particle is initially at rest, and show that

   $$\int d\Pi_2 = \int \frac{d\cos\theta}{4\pi}\frac{p}{2M} \tag{1}$$

   (b) Evaluate the momentum transfer $|\vec{q}|$ as a function of the scattering angle $\cos\theta$.

   (c) Draw the one contributing Feynman diagram. Evaluate this diagram in the non-relativistic limit of the spinors $u(p)$. Use this matrix element in the formula for the differential cross section for electron scattering from a nucleus of charge Z, and derive the Rutherford formula

   $$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2 Z^2}{2m^2\beta^4\sin^4(\theta/2)}\ , \tag{2}$$

   where m is the mass of the electron, $\beta = p/E \approx p/m$ is the velocity of the electron, and $\alpha = e^2/4\pi$.

   (d) Redo the calculation of this scattering cross section using fully relativistic formulae for the u and $\overline{u}$ of the electron (still assuming that the target nucleus is very heavy). Compute the differential cross section averaged over the initial-state electron polarization, and derive the Mott formula

   $$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2 Z^2}{2E^2\beta^4\sin^4(\theta/2)}(1 - \beta^2\sin^2(\theta/2))\ . \tag{3}$$

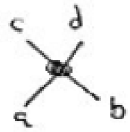
2. Consider the theory of N scalar fields $\varphi^i(x)$ with the Hamiltonian

   $$H = \int d^3x\left(\frac{1}{2}(\Pi^a)^2 + \frac{1}{2}(\vec{\nabla}\varphi^a)^2 + V(\varphi)\right) \tag{4}$$

   with $a = 1, \cdots, N$; repeated indices are summed over.

   $$V(\varphi) = \frac{1}{2}m^2(\varphi^a)^2 + \frac{\lambda}{4}((\varphi^a)^2)^2\ . \tag{5}$$

(a) Consider first the case $m^2 > 0$. In this case, the minimum of $V(\varphi)$ is at $\varphi^a(x) = 0$. Then consider the term with m^2 as part of H_0, and consider the quartic term as a perturbation. Derive the Feynman rules for this theory. Show that there is one vertex, which depends on the indices of the incoming and outgoing particles:



$$= -2i\lambda \left[\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right] \tag{6}$$

(b) Using this Feynman rule, compute the differential cross sections, to leading order in λ, for the processes

$$\varphi^1\varphi^1 \to \varphi^1\varphi^1 \qquad \varphi^1\varphi^2 \to \varphi^1\varphi^2 \qquad \varphi^1\varphi^1 \to \varphi^2\varphi^2 \tag{7}$$

(c) To the same order, compute the total cross section for each of these scattering processes. Note that, for identical bosons in the final state, you should integrate only over $0 < \cos\theta < 1$. Why?

(d) To the same order, compute the total cross section for $\varphi^1\varphi^1$ scattering, summing over all possible final states.

(e) Now consider the case $m^2 = -\mu^2 < 0$. Show that, in this case, the potential V is minimized at a nonzero value, $|\varphi(x)| = v = \mu/\sqrt{\lambda}$. Notice that there are multiple vacuum states, since the vector φ^a can point in any direction. The rotational symmetry of the original model is said to be *spontaneously broken.*

(f) Choose a particular minimum of the potential, $\varphi = (0, 0, \cdots, 0, v)$ and expand about this point by writing

$$\varphi^a(x) = \pi^a(x) \text{ for } a < N; \qquad \varphi^N = v + \sigma(x) \tag{8}$$

Group the quadratic terms in V into H_0, and consider the cubic and quartic terms as a perturbation. Write out H_0 and show that the new fields π^a have zero mass, while the new field σ has mass equal to $\sqrt{2}\mu$.

(g) Work out the Feynman rules for this theory. Now there are 5 distinct vertices:

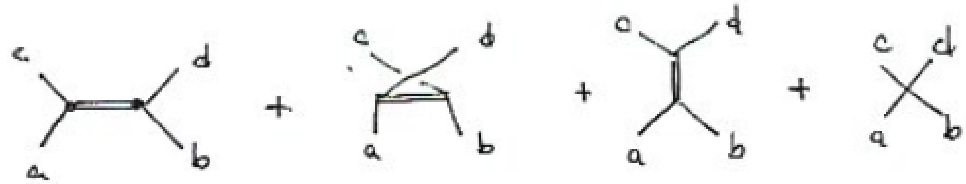
(9)

where the double line denotes a σ propagator and the single lines denote π^a propagators.

(h) Compute the amplitude for the scattering process

$$\pi^a(p_1)\pi^b(p_2) \to \pi^c(p_3)\pi^d(p_4) \tag{10}$$

to the leading order in λ. There are now 4 relevant Feynman diagrams:



(11)

Evaluate these diagrams in terms of the kinematic invariants

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\
t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\
u &= (p_1 - p_4)^2 = (p_2 - p_3)^2
\end{aligned}
\tag{12}
$$

The equalities use $p_1 + p_2 = p_3 + p_4$.

(i) Show that $s + t + u = 0$.

(j) Sum the diagrams and show that sum vanishes at the threshold $s = t = u = 0$.

(k) Show that, in the case $N = 2$, the terms of order p^2 (that is, terms linear in s, t, u, also vanish.

The zeros in found in parts (f), (j), and (k) are not accidents. They result from some deep principles that will be discussed in the later courses in the 330-331-332 sequence.