Physics 330 – Problem Set # 4

(due Thursday, October 23)

1. Let $\phi(x)$ be a free scalar field with mass m, and let $j(x)$ be a c-number function of space-time. Consider the Hamiltonian

$$H = H_0 - g \int d^3x \; j(t, \vec{x})\phi(\vec{x}) \tag{1}$$

The added term allows the source $j(x)$ to create particles from the free-field vacuum. Assume that $j(x) = 0$ in the far past and in the far future. Then the vacuum state in each of these eras is the Fock state vacuum with zero particles.

This theory is solved exactly at the end of Chapter 2 of Peskin and Schroeder. It is shown there that the expectation value for the number n of particles produced is

$$N = \langle n \rangle = g^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \; |\tilde{j}(p)|^2 \; , \tag{2}$$

where $\tilde{j}(p)$ is the Fourier transform of $j(x)$, evaluated at the 4-momentum $p^\mu = (E_p, \vec{p})$. However, you don't have to look at that derivation. We can get the same result from perturbation theory.

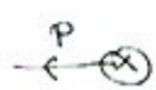
(a) Show that the probability that the source j produces *no* particles is given by

$$P(0) = \left| \langle 0 | \, T \exp[ig \int d^4z \; j(z)\phi_I(z)] \; |0\rangle \right|^2 \tag{3}$$

The vacuum of the perturbed theory is still $|0\rangle$ in the far past and far future, so no denominator is needed. Write a similar expression for the probability for 1 particle of momentum p to be produced, so that the final state is $\langle p|$.

(b) Compute the amplitude for 1 particle of momentum p to be produced, to the leading order in g . This is given by the Feynman diagram with a vertex given by $igj(p)$:

$$ig\tilde{j}(p)$$



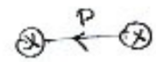
(4)

Sum over final states, using the relativistic sum over states

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \tag{5}$$

and show that, at this order in g, the probability to produce 1 particle is given by (2). This makes sense if the expected number of particles produced is much less than 1, but if N is large, this "probability" can exceed 1, so we need a better formula.

(c) Compute $P(0)$ to order g^2. Show that it is given by

$P(0) = \Big| 1 +$ 

$+ \cdots \Big|^2$ (6)

Evaluate this, and show that $P(0) = 1 - N + \cdots$.

(d) Compute $P(0)$ to all orders in g. This is most easily done by drawing the Feynman diagrams. Show that the diagrams sum to an exponential, giving the result

$$P(0) = \exp[-N] \ . \tag{7}$$

(e) Compute the probability of producing 1 particle to all orders in g. Show that this is

$$P(1) = N \exp[-N] \ . \tag{8}$$

(f) Compute the probability of producing n particles, to all orders in g. Show that this is

$$P(m) = \frac{N^n}{n!} \exp[-N] \ . \tag{9}$$

To get the factor $n!$, remember that the ϕ particles are identical and that a given final state should be counted only once.

(g) The formula (9) is a *Poisson distribution*. Show that

$$\sum_{n=0}^{\infty} P(n) = 1 \qquad \langle n \rangle = \sum_{n=0}^{\infty} n P(n) = N \ . \tag{10}$$

The final state produced by $j(p)$, which is written more explicitly in the exact treatment, is called a *coherent state*. The distribution of particle number in a coherent state is always given by a Poisson distribution.

(h) An interesting case is that in which the source is a point particle that suddenly changes its momentum:

$$j(x) = \int_{-\infty}^{\infty} d\tau \ \delta^{(4)}(x - x(\tau)) \tag{11}$$

where

$$x^\mu(\tau) = \begin{cases} k'^\mu \tau & \tau > 0 \\ k^\mu \tau & \tau < 0 \end{cases} \tag{12}$$

Find the formula for N in this case. (Regulate the integral over τ by cutting it off with a factor $e^{-\epsilon\tau}$ as $|\tau| \to \infty$.)

(i) For the source in (h), show that, if the scalar field has zero mass, the number of particles radiated is infinite. However, most of these particles are concentrated at low values of the energy and momentum. Show that the total energy radiated is finite. Show that bursts of radiation are aligned with $\vec{k}$ and $\vec{k}'$.