Physics 330 – Problem Set # 3

(due Thursday, October 16)

1. The symmetries P, C, and T can have direct implications for particle reactions. Many of these are apparent in thinking about positronium, the bound state of an electron and its antiparticle, the positron. This is a very conventional hydrogenic bound state with reduced mass $\mu = m_e/2$, where m_e is the mass of the electron (or positron). Remember that the electron and positron both have spin, so the $n = 1$ level of positronium has 4 states (1 $S = 0$ state and 3 $S = 1$ states), the $n = 2$ level has 16 states, etc. Remember also that a fermion and its antifermion have the opposite P.

   (a) Write the wavefunction of a positronium state of zero momentum schematically in the form
   $$\int \frac{d^3p}{2\pi}\ \psi(\vec{p})\mathcal{S}_{s_1 s_2}\ a_p^{s_1\dagger}\, b_{-p}^{s_2\dagger}\ |0\rangle\ , \qquad (1)$$
   where $\psi(\vec{p})$ is the spatial wavefunction (in momentum space) and $\mathcal{S}_{s_1 s_2}$ is the spin wavefunction. By thinking about the symmetries of the various factors, argue that such a state with orbital angular momentum L and spin angular momentum $S = 0$ or 1 has
   $$P = (-1)^{L+1} \quad \text{and} \quad C = (-1)^{L+S} \qquad (2)$$

   (b) For a hydrogenic bound state, J is the total angular momentum, considering orbital and spin angular momentum and the fine structure interaction. Find J^{PC} for each state of positronium up to the $n = 3$ level (36 states). For example, the ground state, the 1S state with $S = 0$, has $J^{PC} = 0^{-+}$.

   (c) A photon has vector polarization $\vec{\epsilon}(p)$, so emission of a photon must give $\Delta L = 1$ (electric dipole transition) or $\Delta S = 1$ (magnetic dipole transition). Also, a photon has $C = -1$, since its coupling reverses under C. Work out which of the states in (b) can decay to which others by 1-photon transitions.

   (d) Argue that the $S = 0$ 1S state of positronium can decay to 2 photons, but the $S = 1$ 1S state cannot (and so must decay to 3 photons). This produces a remarkable difference in the lifetimes: $\tau = 1.2 \times 10^{-10}$ sec for $S = 0$, 1.4×10^{-7} sec for $S = 1$.

2. For each of the 16 Dirac bilinears, work out the tranformation properties under P, T, and C. Thus, verify the following table in Peskin and Schroeder's book:

	$\overline{\psi}\psi$	$i\overline{\psi}\gamma^5\psi$	$\overline{\psi}\gamma^\mu\psi$	$\overline{\psi}\gamma^\mu\gamma^5\psi$	$\overline{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
PCT	$+1$	$+1$	-1	-1	$+1$	-1

where $(-1)^\mu = 1$ for $\mu = 0$ and $(-1)^\mu = -1$ for $\mu = 1, 2, 3$. Notice that this table implies that any Lorentz-invariant product of fermion bilinears has $PCT = +1$.

3. The spinor representation of the Lorentz group

$$S^{0i} = i\begin{pmatrix} -\sigma^i/2 & 0 \\ 0 & \sigma^i/2 \end{pmatrix} \qquad S^{ij} = \epsilon^{ijk}\begin{pmatrix} \sigma^k/2 & 0 \\ 0 & \sigma^k/2 \end{pmatrix} \tag{3}$$

is a *reducible* representation. That is, it is the direct sum of two independent representations, one of which acts on the upper two components of a spinor ψ, the other of which acts on the lower two components. In principle, we could build a quantum theory of fermions using one of these representations and not the other. This problem explores that idea.

(a) Start from the massless Dirac theory, with Hamiltonian

$$H = \int d^3x \overline{\Psi}(-i\vec{\nabla} \cdot \vec{\gamma})\Psi \ . \tag{4}$$

Let

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{5}$$

and write this out using our basis of Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{6}$$

Show that H completely splits into a sum of Hamiltonians

$$H = H_L + H_R \ , \tag{7}$$

where H_L includes only ψ_L and H_R includes only ψ_R.

(b) From the canonical commutation relations for the Dirac theory, we can see that the canonical commutation relations for ψ_L and ψ_R are

$$\{\psi_{La}(\vec{x}), \psi^\dagger_{Lb}(\vec{y})\} = \delta_{ab}\,\delta(\vec{x} - \vec{y}) \qquad \{\psi_{Ra}(\vec{x}), \psi^\dagger_{Rb}(\vec{y})\} = \delta_{ab}\,\delta(\vec{x} - \vec{y}) \ . \tag{8}$$

From these formulae and the Hamiltonians H_L, H_R, find the wave equations satisfied by ψ_L and ψ_R. These are called the Weyl equations.

(c) Simplifying the Lorentz transformation laws for ψ_L and ψ_R, show that, under infinitesimal rotations by $\vec{\theta}$ and boosts by $\vec{\eta}$,

$$\begin{aligned} \psi_L &\to (1 - i\vec{\theta} \cdot \vec{\sigma}/2 - \vec{\eta} \cdot \vec{\sigma}/2)\psi_L \\ \psi_R &\to (1 - i\vec{\theta} \cdot \vec{\sigma}/2 + \vec{\eta} \cdot \vec{\sigma}/2)\psi_L \end{aligned} \tag{9}$$

Show that

$$\psi'_L = -i\sigma^2\psi^*_R \quad \text{or} \quad \psi'_{La} = -\epsilon_{ab}\psi^*_{Rb} \tag{10}$$

has the same transformation law as ψ_L. Then ψ_L and ψ_R belong to complex conjugate representations.

(d) Using (10), eliminate ψ_R from H_R in favor of ψ'_L. Show that H_R becomes a second copy of H_L. (You will need to use the fact that the ψ_R and ψ^*_R are *anticommuting* fields.)

What is going on here? ψ_L destroys left-handed fermions and creates right-handed antifermions. ψ_R destroys right-handed fermions and creates left-handed antifermions. So the conjugate of ψ_R creates right-handed fermions and destroys left-handed antifermions. In general, whenever ψ_R appears in our theory, we can reverse the particles and antiparticles and write the theory in terms of a corresponding ψ_L.

Now we can write the most general theory of massless spin-1/2 particles in 4 dimensions,

$$H = \sum_j \int d^3x \Big\{ \psi^\dagger_{jL} (i\vec{\nabla} \cdot \vec{\sigma}) \psi_{jL} \Big\} \ . \tag{11}$$

(e) In the Dirac fermion case, we make the fermion massive by adding to H in (4) the term

$$\Delta H = \int d^3x \ m \overline{\Psi} \, \Psi \ , \tag{12}$$

in which the integrand is Lorentz-invariant. Rewrite this in terms of ψ_L and ψ'_L. You should find

$$\Delta H = \int d^3x \ \ m[\psi^*_{La} \epsilon_{ab} \psi'^*_{Lb} - \psi_{La} \epsilon_{ab} \psi'_{Lb}] \ . \tag{13}$$

Show that the expression $\psi_{La}\epsilon_{ab}\psi'_{Lb}$ is Lorentz-invariant and is symmetric under $\psi \leftrightarrow \psi'$.

(f) The most general massive free-fermion theory in 4 dimensions is then

$$H = \int d^3x \Big\{ \psi^\dagger_{jL} (i\vec{\nabla} \cdot \vec{\sigma}) \psi_{jL} - \frac{1}{2} M_{jk} [\psi_{jLa} \epsilon_{ab} \psi_{kLb}] + \frac{1}{2} M^*_{jk} [\psi^*_{jLa} \epsilon_{ab} \psi^*_{kLb}] \Big\} \ , \tag{14}$$

where M_{jk} can be a general complex-valued symmetric matrix.

Notice that (14) makes sense with a nonzero mass even if there is only 1 left-handed fermion field. Then the matrix M_{jk} reduces to a single number M. This type of mass term is called a Majorana mass. Work out the equation of motion for ψ_L in that case, called the Majorana equation. Show that a solution of the Majorana equation solves the Klein-Gordon equation with mass M.