Physics 330 – Problem Set # 2

(due Thursday, October 9)

1. For applications, it is useful to have explicit forms for the spinors $u(p)$ and $v(p)$. You need to be careful about this, because the product of two non-commuting Lorentz tranformation includes a rotation. For definiteness, we might quantize the spin along the axis of the direction of motion. Define the helicity of a particle as

$$h = \hat{p} \cdot \vec{S} \tag{1}$$

where $\hat{p}$ is a unit vector in the direction of motion. A spin-1/2 particle has 2 helicity states, which we can call "right-handed" and "left-handed", that is $h = +1/2$ and $h = -1/2$.

In discussions of the scattering of Dirac particles, we often need the $u(p)$ and $v(p)$ spinors of definite helicity. Here is a straightforward way to get them. Use the representation of Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} , \qquad \text{with } \sigma^\mu = (1, \vec{\sigma})^\mu \ , \quad \overline{\sigma}^\mu = (1, -\vec{\sigma})^\mu \ . \tag{2}$$

(a) Begin with $u(p)$ for p at rest: $p = (m, 0, 0, 0)$. Then

$$u_\pm(p) = \begin{pmatrix} \sqrt{m}\,\xi_\pm \\ \sqrt{m}\,\xi_\pm \end{pmatrix} \text{, with } \xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{3}$$

Show, using the generators of angular momentum, that $u_+(p)$ has angular momentum $J^3 = +1/2$ and $u_-(p)$ has $J^3 = -1/2$.

(b) Boost $u_+(p)$ along the $\hat{3}$ axis to a momentum p using the explicit expressions for boosts. Show that the result is

$$u_R = \begin{pmatrix} \sqrt{E-p}\ \xi_+ \\ \sqrt{E+p}\ \xi_+ \end{pmatrix} \tag{4}$$

Show that this spinor has $J^3 = +1/2$, that is, it has right-handed spin orientation or right-handed helicity. Show, similarly, that the boost of $u_-(p)$ is

$$u_L = \begin{pmatrix} \sqrt{E+p}\ \xi_- \\ \sqrt{E-p}\ \xi_- \end{pmatrix} \tag{5}$$

and that this is a spinor with $J^3 = -1/2$ or negative helicity.

(c) Now construct the spinors for $p = (E_p, p\sin\theta, 0, p\cos\theta)$. To find these, start from the right- and left-handed spinors in (b) and rotate through an angle θ in the $\hat{3}$-$\hat{1}$ plane. The rotation will use the rotation operators for spinors; thus, angles $\theta/2$ will appear. These spinors have angular momentum +1/2 or -1/2 along the new axis of motion; thus, they are right- and left-handed helicity spinors.

(d) Check explicitly that these rotated spinors satisfy the Dirac equation

$$(\not{p} - m)u(p) = 0 \ . \tag{6}$$

(e) Finally, rotate by ϕ about the $\hat{3}$ axis to construct the spinors of definite helicity about an arbitrary axis.

2. Prove the Gordon identity

$$\overline{u}(p')\gamma^\mu u(p) = \overline{u}(p')\Big[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\Big]u(p) \ , \tag{7}$$

where $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$. We will need this identity later in the course. Hint: $u(p) = (\not{p}/m)u(p)$.

3. In quantum mechanics, we couple the electromagnetic field to the Schrödinger wave-function by the "minimal coupling" prescription

$$\vec{\nabla} \to \vec{\nabla} - ie\vec{A} \tag{8}$$

Let's apply this to the Dirac equation. Write the Dirac equation coupled to electromagnetism as

$$(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi = 0 \tag{9}$$

(a) Compute the square of the Dirac operator

$$(i\gamma^\mu(\partial_\mu + ieA_\mu) + m)(i\gamma^\nu(\partial_\nu + ieA_\nu) - m)\psi = 0 \tag{10}$$

Notice that the result is not the Klein-Gordon equation coupled to electromagnetism; there is an extra term. What is it?

(b) For a Dirac particle at rest, the extra term contributes an energy shift in an external magnetic field proportional to the particle spin. This shift is usually written

$$\Delta E = -\vec{B}\cdot\vec{\mu} \tag{11}$$

where μ is the magnetic moment

$$\vec{\mu} = g\frac{q}{2m}\vec{S} \ , \tag{12}$$

where $\vec{S}$ is the particle spin, q is the electric charge, and g is called the Landé g-factor. For a classical rotating ring of charge, $g = 1$. Show that a Dirac particle has

$$g = 2 \ . \tag{13}$$