Physics 330 – Problem Set # 1

(due Thursday, October 2)

1. In class, we quantized the free scalar field theory with a real-valued scalar field. Now consider the free scalar field theory with with a complex-valued scalar field. This is a theory whose Heisenberg equation of motion is

$$(\partial^2 + m^2)\Phi(x) = 0 \ . \tag{1}$$

where $\Phi(x)$ is a complex-valued operator, that is $\Phi^\dagger(x) \neq \Phi(x)$. This theory has double the number of degrees of freedom of the theory of a real-valued scalar field.

(a) The quantum field Φ, its conjugate $\Phi^\dagger$, and their conjugate momenta Π, $\Pi^\dagger$ can be defined to have the canonical commutation relations

$$\begin{aligned} [\Phi(\vec{x}), \Pi^\dagger(\vec{y})] &= [\Phi^\dagger(\vec{x}), \Pi(\vec{y})] = i\delta(\vec{x} - \vec{y}) \\ [\Phi(\vec{x}), \Pi(\vec{y})] &= [\Phi^\dagger(\vec{x}), \Pi^\dagger(\vec{y})] = 0 \end{aligned} \tag{2}$$

Note carefully that I have defined $\Pi^\dagger$ as the momentum with nonzero commutator with Φ and zero commutator with $\Phi^\dagger$, and vice verse for Π. Please stick to this convention in working this problem set. (You might see the opposite convention elsewhere.)

Using these commutation relations, show that the Hamiltonian

$$H = \int d^3x \Big\{ \Pi^\dagger \Pi + \vec{\nabla}\Phi^\dagger \cdot \vec{\nabla}\Phi + m^2 \Phi^\dagger \Phi \Big\} \tag{3}$$

gives (1) for the equation of motion of $\Phi(x)$ and the conjugate of that equation for the equation of motion of $\Phi^\dagger(x)$.

(b) Show that the formulae

$$\Phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2E_p}} (a_p + b_{-p}^\dagger) \qquad \Phi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2E_p}} (b_p + a_{-p}^\dagger) \ , \tag{4}$$

where $E_p = (|\vec{p}|^2 + m^2)^{1/2}$, with appropriate formulae for Π and $\Pi^\dagger$, and with

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta(\vec{p} - \vec{q}) = [b_p, b_q^\dagger] \tag{5}$$

and a_p, $a_p^\dagger$ commuting with b_p, $b_p^\dagger$, give a representation of the commutation relations (2).

(c) Show that, using (4), the Hamiltonian (3) takes the form

$$H = \int \frac{d^3p}{(2\pi)^3}\; E_p\; (a_p^\dagger a_p + b_p^\dagger b_p)\;, \tag{6}$$

plus an overall constant.

(d) Show that the quantity

$$Q = \int d^3x\; (-i)\Big\{ \Pi^\dagger \Phi - \Phi^\dagger \Pi \Big\} \tag{7}$$

commutes with the Hamiltonian (3). Write out Q in terms of the a_p and b_p operators.

(e) Show that

$$\begin{aligned} [Q, a_p^\dagger] &= (+1)\, a_p^\dagger \qquad & [Q, b_p^\dagger] &= (-1)\, b_p^\dagger \\ [Q, a_p] &= (-1)\, a_p \qquad & [Q, b_p] &= (+1)\, b_p \end{aligned} \tag{8}$$

Show that this implies: the operator $a_p^\dagger$ applied to a quantum state raises the eigenvalue of Q by 1 unit, the operator $b_p^\dagger$ applied to a quantum state lowers the eigenvalue of Q by 1 unit, the operator a_p applied to a quantum state lowers the eigenvalue of Q by 1 unit, the operator b_p applied to a quantum state raises the eigenvalue of Q by 1 unit.

(f) The interpretation of these equations gives very important intuition. We can view the particles created by $\mathbf{a}^\dagger$ and $\mathbf{b}^\dagger$ as two different species of particles. But, these species are related: The field $\Phi(x)$ annihilates the particles of the first type and creates particles of the second type. The conjugate field $\Phi^\dagger(x)$ annihilates particles of the second type and creates particles of the first type. And these species of particle have exactly the same mass. We refer to the two species as particle and antiparticle. The antiparticle has the same mass as the particle and the opposite value of the charge Q.

2. Now consider the case of n real scalar fields.

(a) Consider a theory of n scalar fields with mass m with basic operators $\phi_j(\vec{x})$ and $\pi_j(\vec{x})$, $j = 1, \cdots, n$, satisfying $[\phi_j(\vec{x}), \pi_k(\vec{y})] = i\delta_{jk}\delta(\vec{x} - \vec{y})$. Show that the Hamiltonian

$$H = \sum_j \int d^3x \Big\{ \frac{1}{2}(\pi_j)^2 + \frac{1}{2}(\vec{\nabla}\phi_j)^2 + \frac{1}{2} m^2 \phi_j^2 \Big\} \tag{9}$$

leads to n independent Klein-Gordon equations for the n ϕ_j fields.

(b) Write out H in terms of the creation and annihilation operators a_{jp}, $a_{jp}^\dagger$ for $j = 1, \cdots, n$.

(c) Show that the operators

$$Q_{jk} = \int d^3x \Big\{ -\pi_j \phi_k + \pi_k \phi_j \Big\} \tag{10}$$

are Hermitian and commute with the Hamiltonian (9) for $j, k = 1, \ldots n$. Note that Q_{jj} is zero, and ϕ_j, π_k commute for $j \neq k$.

(d) Write out Q_{jk} in terms of the creation and annihilation operators a_{jp}, $a^\dagger_{jp}$.

(e) Compute the commutator

$$[Q_{jk}, Q_{\ell m}] \ . \tag{11}$$

(f) For $n = 3$, we can equally well write:

$$\mathcal{Q}_1 = Q_{23}, \qquad \mathcal{Q}_2 = Q_{31}, \qquad \mathcal{Q}_3 = Q_{12}, \tag{12}$$

Write the commutation relations of the $\mathcal{Q}_a$.

(g) What is the interpretation of these operators Q_{jk}?