Physics 330 – Quiz 3

Solutions

1.) to begin, let's review

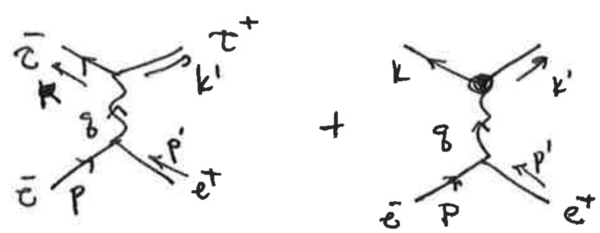
For $e^- e^+ \to \mu^- \mu^+$, in the CM frame, with $m_e = 0$

$$d\sigma = \frac{1}{2E \cdot 2E \cdot 2} \frac{1}{8\pi} \int_{-1}^{1} \frac{d\cos\theta}{2} \left(\frac{k}{E}\right) \frac{1}{4} \sum_{spin} |\mathcal{M}|^2$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2 E_{cm}^2} \frac{1}{16\pi} \sqrt{1 - \frac{4m^2}{s}} \left(\frac{1}{4} \sum_{spin} |\mathcal{M}|^2 \right)$$

This is equally valid for $e^+ e^- \to \tau^+ \tau^-$

$$i\mathcal{M} =$$



$\leftarrow$ a vertex

For the original vertex

$$i\mathcal{M} = (-ie)^2 \, \bar{u}(k) \gamma^\mu v(k') \frac{-i}{q^2} \bar{v}(p') \gamma_\mu u(p)$$

$$\frac{1}{4} \sum_{spins} |\bar{v}(p') \gamma_\mu u(p)|^2 = \frac{1}{4} \sum_{spins} \bar{v}(p') \gamma_\mu u(p) \bar{u}(p) \gamma_\lambda v(p')$$

$$= \frac{1}{4} \operatorname{tr}\left[(\not{p}' - m) \gamma_\mu (\not{p} + m) \gamma_\lambda \right]$$

$$= \frac{1}{4}\, 4 \left[p'_\mu p_\lambda + p'_\lambda p_\mu - p'\cdot p\, \eta_{\mu\lambda} - m^2 \eta_{\mu\lambda} \right]$$

$$= \left[p'_\mu p_\lambda + p'_\lambda p_\mu - \eta_{\mu\lambda} (p'\cdot p + m^2) \right] \Big|_{m_e = 0}$$

$$= \left(p'_\mu p_\lambda + p'_\lambda p_\mu - \eta_{\mu\nu}\, p'\cdot p \right)$$

$$\sum_{spin} \left| \bar u(k) \gamma^\mu v(k') \right|^2 = \sum_{spin} \bar u(k) \gamma^\mu v(k')\, \bar v(k') \gamma^\lambda u(k)$$

$$= tr\left[(\not k + m) \gamma^\mu (\not k' - m) \gamma^\lambda \right] \Big|_{m = m_\tau}$$

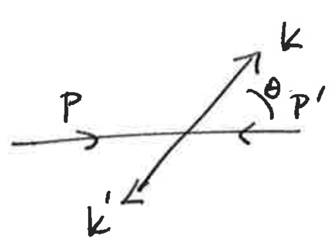
$$= 4 \left[k^\mu k'^\lambda + k^\lambda k'^\mu - \eta^{\mu\lambda} (k\cdot k' + m_\tau^2) \right]$$

the product is

$$4 \left[2 p\cdot k\, p'\cdot k' + 2 p\cdot k'\, p\cdot k - 2\, \cancel{k'\cdot k\, p'\cdot p} - \cancel{2 p'\cdot p}\, k'\cdot k + \cancel{4 p'\cdot p}\, k'\cdot k - 2\, p\cdot p'\, m_\tau^2 + 4\, p\cdot p'\, m_\tau^2 \right]$$

$$= 8 \left[p\cdot k\, p'\cdot k' + p\cdot k'\, p'\cdot k + p\cdot p'\, m_\tau^2 \right]$$

In the kinematics



$p = (E, 0, 0, E)$

$p' = (E, 0\ 0\ -E)$

$k = (E, ks, 0\ kc)$

$k' = (E\ -ks, 0\ -kc)$

$s = \sin\theta$

$c = \cos\theta$

$p\cdot p' = 2E^2$ $\quad p\cdot k = p'\cdot k' = (E^2 - Ekc) = E(E - kc)$

$p\cdot k' = p'\cdot k = E(E + kc)$

this becomes.

$$= 8\,[\;E^2(E-kc)^2 + E^2(E+kc)^2 + 2E^2 m_\tau^2\;]$$

$$= 16E^4[\;1 + \left(\frac{k}{E}\right)^2 c^2 + \frac{m_\tau^2}{E^2}\;] \qquad k^2 = E^2 - m_\tau^2$$

$$= 16E^4[\;1 + c^2 + \frac{m_\tau^2}{E^2} s^2\;]$$

then
$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 = \frac{e^4}{\cancel{E_{cm}^4}} \cdot \cancel{E_{cm}^4}\;[\,1 + c^2 + \frac{m_\tau^2}{E^2} s^2\,]$$

and
$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2E_{cm}^2}\cdot \pi\alpha^2 \sqrt{1-\frac{m_\tau^2}{E^2}}\;\left[1+\cos^2\theta + \frac{m_\tau^2}{E^2}\sin^2\theta\right]$$

Now we have the materials that we need to proceed.

a.) For the complete amplitude

$$i\mathcal{M} = (-ie)^2\left(\frac{-i}{q^2}\right)\;\bar{u}(k)\left(\gamma^\mu + \frac{ia}{2m}\sigma^{\mu\nu}q_\nu\right)v(k')$$
$$\cdot\;\bar{v}(p')\gamma_\mu\, u(p)$$

The only new ingredient we need is $\qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$

$$\sum_{\text{spins}}\left|\,\bar{u}(k)\left(\gamma^\mu + \frac{ia}{2m}\sigma^{\mu\nu}q_\nu\right)v(k')\,\right|^2$$

Let's now compute the $\mathcal{O}(a)$ term in this expression

Note that
$$[\bar{u}(k)(i\sigma^{\mu\nu}q_\nu)v]^* = \left(\bar{u}(k)\left(-\tfrac{1}{2}\right)[\gamma^\mu,\gamma^\nu]\,v\,q_\nu\right)^*$$
$$= \bar{v}\left(-\tfrac{1}{2}[\gamma^\nu,\gamma^\mu]\right)q_\nu\, u(k)$$
$$= \bar{v}\;+\tfrac{1}{2}[\gamma^\mu\gamma^\nu]\,q_\nu\;u(k)$$

so

$\mathcal{O}(a)$ term $= \sum_{\text{spins}} \frac{a}{2m}\Big[\bar u(k)\left(-\tfrac12[\gamma^\mu,\gamma^\nu]q_\nu\right)v(k')\,\bar v(k')\gamma^\lambda u(k)$

$\qquad + \bar u(k)\gamma^\lambda v(k')\,\bar v(k')\left(+\tfrac12[\gamma^\mu\gamma^\nu]q_\nu\right)u(k)\Big]$

$= \frac{a}{2m}\Big\{ \mathrm{tr}\Big[(\not k+m)(-\tfrac12)[\gamma^\mu\not q - \not q\gamma^\mu](\not k'-m)\gamma^\lambda$ $\leftarrow \frac{a}{2m}$ ①

$\qquad + (\not k+m)\,\gamma^\mu(\not k'-m)\,(+\tfrac12)(\gamma^\lambda\not q - \not q\gamma^\lambda)\Big]\Big\}$ $\leftarrow \frac{a}{2m}$ ②

In evaluating traces, note that $\mathrm{tr}[5\gamma\text{'s}] = \mathrm{tr}[3\gamma\text{'s}] = 0$

① $= 4(-\tfrac12)\Big\{m\big[\cancel{q^\mu k'^\lambda} - q^\lambda k'^\mu + \eta^{\mu\lambda}\, q\cdot k'$
$\qquad - \cancel{q^\mu k'^\lambda} + q\cdot k'\,\eta^{\mu\lambda} - q^\lambda k'^\mu\big]$

$\qquad + 4(-m)\big[k^\mu q^\lambda - k\cdot q\,\eta^{\mu\lambda} + \cancel{k^\lambda q^\mu}$
$\qquad - k\cdot q\,\eta^{\mu\lambda} + k^\mu q^\lambda - \cancel{k^\lambda q^\mu}\big]$

$= 4(-\tfrac12)(m)\big[2(k+k')\cdot q\,\eta^{\mu\lambda} - 2q^\lambda(k+k')^\mu\big]$

$= 4(-\tfrac12)\,m\,\big[2(q^2\eta^{\mu\lambda} - q^\mu q^\lambda)\big]$

very nicely, this satisfies  $\cdot\, q^\mu = 0$ current conservation

② $= 4\tfrac12\Big\{m\big[\cancel{k'^\mu q^\lambda} - k'\cdot q\,\eta^{\mu\lambda} + q^\mu k'^\lambda$
$\qquad - \cancel{k'^\mu q^\lambda} + q^\mu k'^\lambda - k'\cdot q\,\eta^{\mu\lambda}\big]$

$\qquad + (-m)\big[\cancel{k^\mu q^\lambda} - k^\lambda q^\mu + k\cdot q\,\eta^{\mu\lambda}$
$\qquad - \cancel{k^\mu q^\lambda} + k\cdot q\,\eta^{\mu\lambda} - k^\lambda q^\mu\big]\Big\}$

$= \tfrac42\, m\,\big[-2(k+k')\cdot q\,\eta^{\mu\lambda} + 2(k+k')^\lambda q^\mu\big]$

$= -\tfrac42\, m\,\big[2(q^2\eta^{\mu\lambda} - q^\mu q^\lambda)\big]$

~~In~~ all

$$O(a) \text{ term} = 4 \frac{a}{2m} (-2m) \left(q^2 \eta^{\mu\lambda} - q^\mu q^\lambda \right)$$

contract this with

$$\left(p'_\mu p_\lambda + p'_\lambda p_\mu - \eta_{\mu\lambda}\, p \cdot p' \right)$$

$$= (-4a) \left[2 q^2\, p \cdot p' - 4 q^2\, p \cdot p' - 2\, p' \cdot q\, p \cdot q + q^2\, p \cdot p' \right]$$

since the electrons are massless $\quad p^2 = p'^2 = 0$

$$p' \cdot q = p'(p+p') = p' \cdot p \qquad q^2 = 2 p \cdot p'$$

then

$$= (-4a) \left[(q^2)^2 - 2 (q^2)^2 - q^2\, p p' + q^2\, p \cdot p' \right]$$

$$= 4 a (q^2)$$

the result, up to this point, is

$$\frac{1}{4} \sum |M|^2 = \frac{e^4}{(q^2)^2} \left\{ q^2 \left[1 + \cos^2\theta + \frac{m_\tau^2}{E^2} \sin^2\theta + 4a \right] \right\}$$

so

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2E_{cm}^2} \pi \alpha^2 \sqrt{1 - \frac{4 m_\tau^2}{E_{cm}^2}} \left[1 + 4a + \cos^2\theta + \frac{m_\tau^2}{E} \sin^2\theta \right]$$

b.) Now work on the $\mathcal{O}(a^2)$ term in $\frac{1}{4}\sum |M|^2$

for this we need

$$\sum_{\text{spins}} \left(\frac{a}{2m}\right)^2 \bar{u}(k)\left(-\tfrac{1}{2}[\gamma^\mu,\gamma^\nu]q_\nu\right) v(k')\bar{v}(k')\left(-\tfrac{1}{2}[\gamma^\lambda,\gamma^\sigma]q_\sigma\right) u(k)$$

$$= \left(\frac{a}{2m}\right)^2 tr\left[(\not{k}+m)(\gamma^\mu \not{q} - \not{q}\gamma^\mu)(\not{k}'-m)(\gamma^\lambda \not{q} - \not{q}\gamma^\lambda)\right]\cdot\left(-\tfrac{1}{4}\right)$$

I advised you to drop the m terms relative to the k terms, which are leading at high energy. Then

$$= \left(-\frac{a^2}{16m^2}\right) tr\big[\ \not{k}\gamma^\mu\not{q}\not{k}'\gamma^\lambda\not{q} \qquad (1)$$
$$- \not{k}\not{q}\gamma^\mu\not{k}'\gamma^\lambda\not{q} \qquad (2)$$
$$- \not{k}\gamma^\mu\not{q}\not{k}'\not{q}\gamma^\lambda \qquad (3)$$
$$+ \not{k}\not{q}\gamma^\mu\not{k}'\not{q}\gamma^\lambda\big] \qquad (4)$$

This is going to get complicated. Be very organized.

Since $\not{q}\not{q} = q^2$ we can simplify these expressions by moving the $\not{q}$'s together. e.g.

$$\not{q}\not{k}\not{q} = 2q\cdot k\,\not{q} - \not{k}q^2$$

then

$$(1) = tr[\not{k}\gamma^\mu\not{q}\not{k}'\gamma^\lambda\not{q}]$$

$$= tr\left[\not{k}\gamma^\mu(2q\cdot k')\gamma^\lambda\not{q} - \not{k}\gamma^\mu\not{k}'(2q^\lambda)\not{q} + \not{k}\gamma^\mu\not{k}'\gamma^\lambda q^2\right]$$

and now we can apply the formula for 4 γ's

$$① = 4\cdot\Big[(2q\cdot k')(k^\mu q^\lambda - k^\lambda q^\mu + k\cdot q\,\eta^{\mu\lambda})$$
$$+ 2q^\lambda(-k^\mu k'\cdot q + k\cdot k' q^\mu - k\cdot q\, k'^\mu)$$
$$+ q^2(k^\mu k'^\lambda - \eta^{\mu\lambda} k\cdot k' + k^\lambda k'^\mu)\Big]$$

$$② = tr[-\not{k}\not{q}\gamma^\mu \not{k}'\gamma^\lambda\not{q}]$$
$$= tr[-(2k\cdot q)\gamma^\mu\not{k}'\gamma^\lambda\not{q} + q^2\not{k}\gamma^\mu\not{k}'\gamma^\lambda]$$
$$= 4\Big[(2k\cdot q)(-k'^\mu q^\lambda + \eta^{\mu\nu}k'\cdot q - k'^\lambda q^\mu)$$
$$+ q^2(k^\mu k'^\lambda - \eta^{\mu\nu}k\cdot k' + k^\lambda k'^\mu)\Big]$$

$$③ = tr[-\not{k}\gamma^\mu\not{q}\not{k}'\not{q}\gamma^\lambda]$$
$$= tr[(+2k'\cdot q)(-\not{k}\gamma^\mu\not{q}\gamma^\lambda) + q^2\not{k}\gamma^\mu\not{k}'\gamma^\lambda]$$
$$= 4\Big[(2k'\cdot q)(-k^\mu q^\lambda + \eta^{\mu\lambda}k\cdot q - k^\lambda q^\mu)$$
$$+ q^2(k^\mu k'^\lambda - \eta^{\mu\lambda}k\cdot k' + k^\lambda k'^\mu)\Big]$$

$$④ = tr[\not{k}\not{q}\gamma^\mu\not{k}'\not{q}\gamma^\lambda]$$
$$= tr[(2q^\mu)\not{k}\not{k}'\not{q}\gamma^\lambda - 2q\cdot k'(\not{k}\gamma^\mu\not{q}\gamma^\lambda) + q^2(\not{k}\gamma^\mu\not{k}'\gamma^\lambda)]$$
$$= 4\Big[(2q^\mu)(k\cdot k' q^\lambda - k\cdot q\, k'^\lambda + k^\lambda q\cdot k')$$
$$+ (2q\cdot k')(-k^\mu q^\lambda + k\cdot q\,\eta^{\mu\lambda} - k^\lambda q^\mu)$$
$$+ q^2(k^\mu k'^\lambda - \eta^{\mu\lambda}k\cdot k' + k^\lambda k'^\mu)\Big]$$

$(1) + (2) + (3) + (4)$

$$= 4 \left[\eta^{\mu\lambda} \left(2k' \cdot q\, k \cdot q - k \cdot k' q^2 + 2k \cdot q\, k' \cdot q - k' \cdot k\, q^2 + 2k' \cdot q\, k \cdot q - k \cdot k' q^2 + 2k \cdot q\, k' \cdot q - k' \cdot k\, q^2 \right) \right.$$
$$+ (k^\mu k'^\lambda + k^\lambda k'^\mu)(q^2 + q^2 + q^2 + q^2)$$
$$+ q^\mu q^\lambda (2k \cdot k' + 2k \cdot k')$$
$$+ k^\mu q^\lambda (2q \cdot k' - 2q \cdot k' - 2q \cdot k' - 2q \cdot k')$$
$$+ q^\mu k^\lambda (-2q \cdot k' - 2q \cdot k' + 2q \cdot k' - 2q \cdot k')$$
$$+ k'^\mu q^\lambda (-2q \cdot k - 2q \cdot k)$$
$$\left. + q^\mu k'^\lambda (-2q \cdot k - 2q \cdot k) \right]$$

$$= 4 \left[\eta^{\mu\lambda} (8 k \cdot q\, k' \cdot q - 4 k \cdot k' q^2) + q^\mu q^\lambda (4 k \cdot k') \right.$$
$$+ (k^\mu k'^\lambda + k^\lambda k'^\mu)(4q^2)$$
$$+ (k^\mu q^\lambda + q^\mu k^\lambda)(-4 q \cdot k')$$
$$\left. + (k'^\mu q^\lambda + q^\mu k'^\lambda)(-4 q \cdot k) \right]$$

Now, if we ignore m_Z and treat k, k' as massless

$$k \cdot k' = \tfrac{1}{2}(k+k')^2 = \tfrac{1}{2} q^2$$

$$k \cdot q = k \cdot (k+k') = k \cdot k' = k'(k+k') = k' \cdot q = \tfrac{1}{2} q^2$$

then the above becomes.

$$= 4 \left[\eta^{\mu\lambda} \left(2(q^2)^2 - 2(q^2)^2 \right) + q^\mu q^\lambda (2q^2) \right.$$
$$+ (k^\mu k'^\lambda + k^\lambda k'^\mu)\, 4q^2$$
$$+ \left((k+k')^\mu q^\lambda + (k+k')^\lambda q^\mu \right)(-2q^2)$$

$$= 4\left[(k^\mu k'^\lambda + k^\lambda k'^\mu)\, 4q^2 + q^\mu q^\lambda (2q^2) + q^\mu q^\lambda(-4q^2) \right]$$

$$= (-4)\left[(k+k')^\mu (k+k')^\lambda \cdot 2q^2 - (k^\mu k'^\lambda + k^\lambda k'^\mu)\, 4q^2 \right]$$

$$= -4 \cdot 2q^2 \left[(k-k')^\mu (k-k')^\lambda \right]$$

$$= -8q^2 \left[(k-k')^\mu (k-k')^\lambda \right]$$

Then

$$\sum_{\text{spins}} |\mathcal{M}|^2 = -\left(\frac{a}{2m}\right)^2 \cdot \frac{1}{4} \; (-8q^2)\, (k-k')^\mu (k-k')^\lambda$$

square of electron current $\rightarrow$ $\cdot\; 4\left[p^\mu p'^\lambda + p^\lambda p'^\mu - \eta^{\mu\nu}\, p\cdot p' \right]$

$$= \frac{a^2}{m^2} \cdot 2q^2 \cdot \left[2p\cdot(k-k')\; p'\cdot(k-k') - (k-k')^2\, p\cdot p' \right]$$

with:

$$p = (E, 0\; 0\; E) \qquad p' = (E, 0\; 0\; -E)$$

$$k = (E,\; E\sin\theta, 0\; E\cos\theta) \qquad k' = (E, -E\sin\theta, 0\; -E\cos\theta)$$

$$k - k' = (0,\; 2E\sin\theta,\; 0,\; 2E\cos\theta)$$

$$p\cdot(k-k') = -2E^2\cos\theta \qquad p'\cdot(k-k') = +2E^2\cos\theta$$

$$(k-k')^2 = -4E^2 \qquad p\cdot p' = 2E^2$$

$$= \frac{a^2}{m^2}\, 2q^2 \cdot (8E^4 - 8E^4\cos^2\theta)$$

then

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 = \frac{1}{4}\frac{a^2}{m^2} \cdot q^2 \cdot (2E)^4 (\sin^2\theta)$$

$$= \frac{a^2}{4m^2} (q^2)^3$$

Then finally, at energies $E_{cm} \gg m_\tau$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2E_{cm}^2} \cdot \left[1 + \cos^2\theta + 4a + \frac{a^2}{4m^2} E_{cm}^2 \sin^2\theta \right]$$

c.) The term in a^2 grows as $\frac{E_{cm}^2}{m^2}$ relative to the leading order cross section. This is actually a problem. The enhancement is all in one partial wave (P-wave). It can be shown that, in each partial wave, the scattering amplitude is bounded by a constant. So this behavior eventually violates partial wave unitarity.

The result also looks a little weird. We have come to expect that the dominant terms in the scattering amplitude at high energy are helicity-conserving. But the a term is completely helicity violating. If a is nonzero, then there are strong helicity-violation τ interactions at $E_{strong} \sim \frac{m_\tau}{a} \gg m_\tau$

But then, why doesn't the τ get a mass of order E_{strong}?

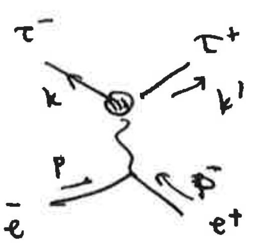
Addendum :

Several people in the class tried to solve this problem using the Gordon identity. This is a little more conceptually complicated than the "plug and chug" method used by most of the class, but it does make the computation much easier. Here is the solution by that method:

We need to compute

$$iM = (-ie)^2 \; \bar{u}(k)\left[\gamma^\mu + a \cdot \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] v(k') \; \frac{-i}{q^2}$$

$$\bar{v}(p') \gamma_\mu u(p)$$



Work out the relevant Gordon identity

$$\bar{u}(k)\gamma^\mu v(k') = \frac{1}{2m}\, \bar{u}(k)\left[\not{k}\gamma^\mu - \gamma^\mu \not{k}' \right] v(k')$$

since $(\not{k}' + m)v(k') = 0$

$$= \frac{1}{2m}\bar{u}(k)\Big[\tfrac{1}{2}\{\not{k}, \gamma^\mu\} - \tfrac{1}{2}\{\gamma^\mu, \not{k}'\}$$

$$+ \tfrac{1}{2}[\not{k}, \gamma^\mu] - \tfrac{1}{2}[\gamma^\mu, \not{k}'] \Big] v(k')$$

$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

$$= \frac{1}{2m}\bar{u}(k)\left[(k-k')^\mu + i\sigma^{\mu\nu}\underbrace{(k'+k)_\nu}_{q} \right] v(k')$$

then

$$\bar{u}(k)\,\frac{i\sigma^{\mu\nu}}{2m}q_\nu\, v(k') = \bar{u}(k)\left(\gamma^\mu - \frac{(k-k')^\mu}{2m}\right) v(k')$$

then the τ matrix element from the above is

$$\mathcal{M}^\mu_\tau = \bar{u}(k)\left[\gamma^\mu(1+a) - a\frac{(k-k')^\mu}{2m}\right] v(k')$$

Square this

$$\sum_{\text{spins}} \mathcal{M}^\mu_\tau\, \mathcal{M}^{*\lambda}_\tau$$

$$= \operatorname{tr}\Big\{(\not{k}+m)\left[\gamma^\mu(1+a) - a\frac{(k-k')^\mu}{2m}\right](\not{k}'-m)$$
$$\left[\gamma^\lambda(1+a) - a\frac{(k-k')^\lambda}{2m}\right]\Big\}$$

There are 4 terms. The first one is what we computed in class

$$\operatorname{tr}\left[(\not{k}+m)\gamma^\mu(1+a)(\not{k}'-m)\gamma^\lambda(1+a)\right]$$

$$= 4\left(k^\mu k'^\lambda + k^\lambda k'^\mu - \eta^{\mu\lambda}k\cdot k' - m^2\eta^{\mu\lambda}\right)(1+2a+a^2)$$

By itself, this would give

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2E_{cm}^2}\sqrt{1-\frac{4m_\tau^2}{E_{cm}^2}}\cdot\left(1+\cos^2\theta + \frac{4m_\tau^2}{E_{cm}^2}\sin^2\theta\right)$$

Now compute the cross terms. Begin with

$$\mathrm{tr}\left\{ (\not k+m)\, \gamma^\mu (1+a)(\not k'-m)\left(\frac{-a}{2m}\right) \frac{(k-k')^\lambda}{2m} \right\}$$

only the term with 2 γ's is nonzero

$$= 4\left(m\, k'^\mu - m\, k^\mu \right)(1+a)\left(\frac{-a}{2m}\right)(k-k')^\lambda$$

$$= 4\,\frac{a}{2}\,(1+a)\,(k-k')^\mu (k-k')^\lambda$$

$$= 2a\,(1+a)\left(k^\mu k^\lambda + k'^\mu k'^\lambda - k^\mu k'^\lambda - k^\lambda k'^\mu\right)$$

the other cross term gives the same result. Add all of the a term

$$= 4a\left[(k-k')^\mu (k-k')^\lambda + 2\left(k^\mu k'^\lambda + k^\lambda k'^\mu\right) - \eta^{\mu\nu}(2\,k\cdot k') \right]$$

$q^2 = 2k\cdot k' + 2m^2$

$$= 4a\left[(k+k')^\mu (k+k')^\lambda - \eta^{\mu\nu} q^2 + \cancel{\eta^{\mu\nu} 2m^2} - \cancel{\eta^{\mu\nu}} 2m^2 \right]$$

$$= -4a\left[q^2 \eta^{\mu\nu} - q^\mu q^\nu \right]$$

exactly the result at the top of p. 5 !

this immediately gives

$$\Delta \frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2E_{cm}^2}\sqrt{1-\frac{4m_t^2}{E_{cm}^2}} \cdot 4a$$

the a^2 term is even easier. At high energy, this term is enhanced by the factor

$$\left(a \frac{(\text{Energy})}{m_\tau} \right)^2$$

None of the other a^2 terms have this dependence, so the leading a^2 behavior is

$$\mathrm{tr}\, (\not{k}+m)\left(- a\frac{(k-k')^\mu}{2m}\right)(\not{k}'-m)\left(- a\frac{(k-k')^\lambda}{2m}\right)$$

$$= 4(k\cdot k')\left(\frac{a}{2m}\right)^2 (k-k')^\mu (k-k')^\lambda$$

$$= 2q^2\left(\frac{a}{2m}\right)^2 (k-k')^\mu (k-k')^\lambda$$

this is exactly the result on p. 9

$$\sum_{\text{spin}} |\mathcal{M}_\tau|^2 = -\left(\frac{a}{2m}\right)^2 \frac{1}{4}(-8q^2)(k-k')^\mu (k-k')^\lambda$$

$$= 2q^2\left(\frac{a}{2m}\right)^2 (k-k')^\mu (k-k')^\lambda$$

then finally, at high energy,

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2E_{cm}^2}\left[1+\cos^2\theta + 4a + \frac{a^2}{4m^2}E_{cm}^2\sin^2\theta\right]$$

!!!