Physics 330 – Final Exam

(due Thursday, December 11)

I will use the quiz and the final in this course to assign grades, keeping in mind that anyone who has handed in all of the problem sets is assured a reasonable grade in the course. Because this exam will be graded, please abide by these rules:

- The quizzes are open-book. You may use any reference resources that you find. You may use your mathematical software such as MatLab or Mathematica. However, please do not collaborate with other students or ask help from any other human—with the exception that, if you have any question about the exam, please feel free to email me (mpeskin@slac.stanford.edu). If you find an AI that can solve this exam, please let me know about it.
- The final is posted at the course web site:

  `https://s3df.slac.stanford.edu/people/mpeskin/Physics330/`

  If there are corrections, the latest version can be found there.
- If you plan not to hand in the exam, this is acceptable, but please email me and let me know. Otherwise I will worry that your exam was lost.
- Please write on your solution: "I acknowledge the Stanford Honor Code." and sign it.
- Please hand in your solution to the final to Gradescope by 11:00pm on Thursday, Dec. 11. This is a sharp deadline. If this poses a problem for you, please email me in advance.

The final will be worth 70 points. (The Quiz was worth 30 points.) Partial credit will be given.

1. (30 points)

In the Standard Model of particle physics, the weak and electromagnetic interactions are mediated by four vector bosons, called W^+, W^-, W^0 and B^0. For this problem, I will simplify the setup a little. Consider a theory of two heavy Dirac fermions Ψ_a, $a = 1, 2$, with mass M coupled to a set of two neutral massive vector bosons, W and B, via the Hamiltonian

$$H_{int} = \int d^3x \left[\frac{g}{2} W_\mu \overline{\Psi}_a \gamma^\mu \sigma^3_{ab} \Psi_b + \frac{g'}{2} B_\mu \overline{\Psi}_a \gamma^\mu \Psi_a \right] \tag{1}$$

For the Feynman propagator of the massive bosons, use

$$\underset{q}{\mu \sim\!\sim\!\sim\!\sim \nu} \quad = \quad \frac{-i\,\eta^{\mu\nu}}{q^2 - m^2} \tag{2}$$

Compute the mass shift of the two states Ψ^a due to exchange of the W and B bosons. Use the parameters of the Standard Model:

$$\alpha_w = g^2/4\pi = 1/30 \qquad \alpha' = (g')^2/4\pi = 1/99 \qquad v = 246 \text{ GeV} \tag{3}$$

Some formulae in this problem are analous to formulae in the Standard Model. But, you should not have to read up on the Standard Model to solve this problem. People who have already studied the Standard Model will recognize the equations (4)–(7). I write these equations explicitly to bring the rest of you up to speed.

(a) First, assume that the W boson and the B boson have masses

$$m_W = \frac{gv}{2} \qquad m_B = \frac{g'v}{2} \tag{4}$$

Compute the mass shifts of Ψ_1 and Ψ_2 from the bare value M. Show that these are log divergent but equal if the diagrams are regularized in the same way. You can use either Pauli-Villars or dimensional regularization. You need only compute the coefficient of $\log \Lambda^2$.

(b) Next, assume, in analogy to the Standard Model, that the vector bosons (W, B) have the mass matrix

$$m^2 = \begin{pmatrix} g^2v^2/4 & -gg'v^2/4 \\ -gg'v^2/4 & g'^2v^2/4 \end{pmatrix} \tag{5}$$

Diagonalize this matrix and find its eigenvectors. Notice that one of the eigenvectors has $m^2 = 0$. In the real model, this would be identified with the photon. The photon field has an expression analogous to that in the Standard Model:

$$A_\mu = \cos\theta\, B_\mu + \sin\theta\, W_\mu \tag{6}$$

with

$$\sin\theta = \frac{g'}{\sqrt{g^2 + g'^2}} \qquad \cos\theta = \frac{g}{\sqrt{g^2 + g'^2}} \tag{7}$$

(c) Rewrite eq. (1) in terms of the mass eigenstates. Notice that Ψ_2 now has zero electric charge with respect to the photon. Then Ψ_1 should get a positive mass shift from its electromagnetic interaction that is not present for Ψ_2.

(d) Compute the mass shifts for Ψ_1 and Ψ_2. Show that they are ultraviolet divergent. However, show that the mass *difference* is finite.

(e) Compute the mass difference between Ψ_1 and Ψ_2 for $M \gg m_W$. Show that the difference is independent of the value of M in this limit.

(f) Using the parameters above, obtain an explicit value (in MeV) for the mass difference in the $M/m_W \to \infty$ limit.

2. (40 points)

Out in the universe, 85% of the mass is in the form of dark matter, made of particles that are neutral and very weakly interacting. In principle, we can produce these particles in collider experiments and learn their properties from measurements of the final state in reactions that produce them. Let S be a complex scalar particle that is a candidate for the the dark matter particle, and let F be a Dirac fermion that is a candidate for the dark matter particle. Assume that the interactions of S and F with ordinary matter are so weak that these particles make no signal in standard particle detectors.

Of course the reactions $e^+e^- \to S\,\overline{S}$ and $e^+e^- \to F\,\overline{F}$ are not observable at a collider. But, in the reactions with photon emission

$$e^+e^- \to S\,\overline{S} + \gamma\ , \qquad e^+e^- \to F\,\overline{F} + \gamma\ , \tag{8}$$

the energy spectrum of the final-state photon can tell us something about the invisible particles. In this problem, we will compare the photon spectra in models with an S or F, in each case with mass M, as the dark matter particle and study how to distinguish these models.

To be concrete, consider a model in which there is a very heavy vector boson B that couples to the currents of the S, the F, and the electron. The Feynman rules for the B interactions are

$$S\ (p \to p'):\ = -ig(p+p')^\mu \qquad F:\ = -ig\gamma^\mu \qquad e:\ = -ig\gamma^\mu \tag{9}$$

Take the B propagator to be

$$\mu \overset{q}{\sim\sim} \nu \ = \ \frac{-i\eta^{\mu\nu}}{q^2 - M_B^2} \tag{10}$$

It is useful to consider center of mass energies E_{CM} that are greater than but comparable to $2M$. Set the electron mass to 0 in this problem.

(a) In (9), the vertices for the electron and the F are current couplings with the same form as the electromagnetic current vertex in QED. Then they satisfy $q^\mu \tilde{j}_\mu(q) = 0$ when the external fermions are on shell. Show that the vertex for S is also a proper current vertex by showing that $q^\mu \tilde{j}_\mu(q) = 0$ also for this case, when the external scalars S are on shell.

(b) Draw the Feynman diagrams for the processes with S and F pair production and photon emission written in (8).

(c) Write an expression for the scattering matrix element in each process.

(d) Work in the limit $M_B \gg E_{CM}$. Show that, in this limit , the cross section depends only on the ratio g^2/M_B^2.

(e) There are 3 particles in the final state, so we will need a representation of 3 body phase space. Actually, the situation here is easier than in the general case. Let p, $\overline{p}$ be the final momenta of the dark matter particles, and let $P = p + \overline{p}$; let k be the final momentum of the photon. Rewrite the expression for 3-body phase space as an integral over the 2-body phase space of $(p, \overline{p})$ followed by an integral over the 2-body phase space of (k, P).

(f) Argue that the vector that equals $(1, 0, 0, 0)^\mu$ in the P rest frame can also be written as $P^\mu/\sqrt{P^2}$, a representation valid in any frame. Find a similar representation that is valid in any frame for the tensor that equals

$$\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{\mu\nu} \tag{11}$$

in the P rest frame.

(g) Calculate the cross section for the F pair production process, for unpolarized electron and positron beams. The easiest way to do this is to carry out the various operations in the following order: (1) compute the trace involving the (unobservable) F and $\overline{F}$ momenta, (2) carry out the phase space integral over the $(p, \overline{p})$ phase space in the P rest frame, (3) rewrite this result in a general frame, (4) compute the trace over the electron and positron spinors, using the result of (3). You should end up with a differential cross section

$$\frac{d\sigma}{dk\, d\cos\theta}\ , \tag{12}$$

where k is the photon momentum and $\cos\theta$ is the angle between the photon momentum and the e^- beam direction.

(h) Calculate the cross section for the S pair production process, using the same steps as in part (g) (and taking advantage of the intuition you have gained).

(i) Plot the differential cross section (12) as a function of k for each reaction at several fixed values of $\cos\theta$.

(j) Explain how to measure the dark matter particle mass M.

(k) Explain how to distinguish the F and S hypotheses.