Physics 330 – Final Exam

(due Wednesday, December 14)

I will use the quiz and the final in this course to assign grades, keeping in mind that anyone who has handed in all of the problem sets is assured a reasonable grade in the course. Because this exam will be graded, please abide by these rules:

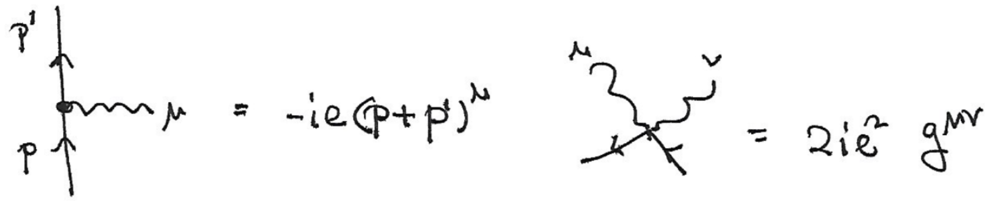
- The quizzes are open-book. You may use any reference resources that you find. You may use your mathematical software such as MatLab or Mathematica. However, please do not collaborate with other students or ask help from any other human—with the exception that, if you have any question about the exam, please feel free to email me (mpeskin@slac.stanford.edu). If you find an AI that can solve this exam, please let me know about it.
- The final is posted at the course web site:

  `https://www.slac.stanford.edu/~mpeskin/Physics330/`

  If there are corrections, the latest version can be found there.
- If you plan not to hand in the exam, this is acceptable, but please email me and let me know. Otherwise I will worry that your exam was lost.
- Please write on your solution: "I acknowledge the Stanford Honor Code." and sign it.
- Please hand in your solution to the final to Gradescope by 4:00pm on Wednesday, Dec. 11. This is a sharp deadline. If this poses a problem for you, please email me in advance.

The final will be worth 70 points. (The Quiz was worth 30 points.) Partial credit will be given.

1. (20 points) Consider the electromagnetic interactions of a complex-valued scalar field Φ of charge $(-e)$ and mass m. The Feynman rules for the scalar-photon interaction are



(1)

These rules can be derived from the Lagrangian for this system

$$L = \int d^3x \left[(D^\mu \Phi)^* D_\mu \Phi - m^2 \Phi^* \Phi \right] , \tag{2}$$

where $D_\mu = (\partial_\mu + ieA_\mu)$. This Lagrangian motivates the vertex with two photons, which does not appear for Dirac fermions.

(a) Let $V^\mu(p', p, q)$ be the vertex with one photon. Since the scalar field current should be conserved, we expect that

$$q_\mu V^\mu(p', p, q) = 0 \tag{3}$$

if the two scalars are on mass shell. Show this explicitly.

(b) Compute the differential cross section $d\sigma/d\cos\theta$ for the process $e^+e^- \to \Phi^+\Phi^-$ in the center of mass system, and integrate this to find the total cross section. As we did for $e^+e^- \to \mu^+\mu^-$, set the electron mass to 0 and average over the initial electron spins. Compare your answers to the results for $e^+e^- \to \mu^+\mu^-$ discussed in class, and argue that the spin 0 and spin 1/2 cases are easily distinguished experimentally.

(c) Draw the Feynman diagrams for the reaction $\gamma\gamma \to \Phi^+\Phi^-$ with initial momenta k_1, k_2 and final momenta p , $\bar{p}$.

(d) The scattering amplitude for this process has the form

$$i\mathcal{M} = (-ie)^2 M^{\mu\nu}\, \epsilon_{1\mu}\epsilon_{2\nu} \ , \tag{4}$$

where $\epsilon^\mu_{1,2}$ are the polarization vectors of the two photons. Show explicitly that the scalar field current is conserved:

$$k_{1\mu} M^{\mu\nu}\, \epsilon_{2\nu} = 0 \tag{5}$$

if the external scalar particles are on shell.

2. (50 points) In the 1940's, physicists tried to describe the nucleon-nucleon interaction by the exchange of π mesons. Two possible forms of the nucleon-pion coupling were the "pseudoscalar" and "pseudovector" coupling. Let me explain these.

The neutron and proton form a doublet of fields

$$N = \begin{pmatrix} p \\ n \end{pmatrix} \tag{6}$$

where p and n are the Dirac fermion fields of the proton and neutron. Since the masses of the proton and neutron are almost equal, it is suggestive that the strong interactions have a symmetry that interchanges p and n. Heisenberg suggested that the strong interactions are approximately invariant under $SU(2)$ rotations, called "isospin", that act on this doublet as an $SU(2)$ spinor. In fact, this symmetry is valid to an accuracy of a few percent for simple nuclei, despite the fact that the proton has electric charge and the neutron does not.

There are 3 π mesons, which form a 3-vector under $SU(2)$ isospin. These mesons also have odd parity, that is, for the quantum field $\pi^i(x)$, $P\pi^i(x)P = -\pi^i(x)$. It is useful to define pion field of definite electric charge by

$$\pi^+ = \frac{1}{\sqrt{2}}(\pi^1 + i\pi^2) \qquad \pi^- = \frac{1}{\sqrt{2}}(\pi^1 - i\pi^2) \qquad \pi^0 = \pi^3 \tag{7}$$

(a) A pion interaction with the isospin-1/2 nucleons can then be written in terms of real-valued pion fields as

$$\pi^i\sigma^i = \pi^1\sigma^1 + \pi^2\sigma^2 + \pi^3\sigma^3 \ , \tag{8}$$

here σ^i are the Pauli sigma matrices, or as

$$\sqrt{2}\ \pi^+\sigma^- + \sqrt{2}\ \pi^-\sigma^+ + \pi^0\sigma^3 \tag{9}$$

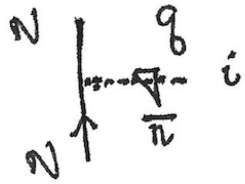
Show that (8) and (9) are equal. The π^- field creates a π^- particle while turning an n into an p. Its conjugate π^+ creates a π^+ or destroys a π^- particle while turning a p into an n.

In a completely isospin-symmetric theory, the three pions would have the same mass m_π and the proton and neutron would have the same mass m_N. Assume this while doing the rest of this problem.

(b) A form of the nucleon-pion interaction that is invariant under Lorentz invariance, parity, and isospin is

$$\Delta H = \int d^3x \left(ig_{\pi NN}\ \pi^i\ \overline{N}\sigma^i\gamma^5 N \right) \ , \tag{10}$$

leading to the Feynman rule



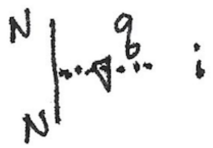
$$= g_{\pi NN}\, \sigma^i \gamma^5 \tag{11}$$

This is called the "pseudoscalar theory".

There was another proposed interaction called the "pseudovector theory",

$$\Delta H = \int d^3x \left(-\frac{g_{\pi NN}}{2m_N}\, \partial_\mu \pi^i\, \overline{N} \sigma^i \gamma^\mu \gamma^5 N \right) , \tag{12}$$

leading to the Feynman rule



$$= \frac{g_{\pi NN}}{2m_N}\, q_\mu \gamma^\mu \gamma^5 \sigma^i \tag{13}$$

The exercise in the rest of this problem is to compare these two theories.

Show that the Dirac equation implies

$$\partial_\mu \overline{N} \sigma^i \gamma^\mu \gamma^5 N = 2im_N \overline{N} \sigma^i \gamma^5 N \ . \tag{14}$$

Thus, these theories might be expected to be equivalent. However, there might be surprises from short-distance or large-q contributions.

(c) Work out the leading term for the nucleon-nucleon scattering amplitude in the pseudoscalar theory in the limit in which the nucleons are heavy and moving slowly. Use the representation (8). For simplicity, treat the nucleons as distinguishable; then there is only 1 Feynman diagram. Show that this takes the form of an interaction that couples the spins of the nucleons and also depends on their isospins. To obtain a nonzero result, you will need to expand the formulae for the spinors $u(p)$ and $\overline{u}(p')$ to order p^1. You will find that the interaction vanishes for the exchanged momentum $q = 0$.

For a successful theory of nuclear forces, there should be a contribution that is a universally attractive potential between any two nucleons. So this theory, by itself, is not successful; additional exchanges need to be added. Nevertheless, proceed.

(d) Work out the leading term for the nucleon-nucleon scattering amplitude in the pseudovector theory in the limit in which the nucleons are heavy and moving slowly. For simplicity, treat the nucleons as distinguishable. Show that this is identical to the result for the pseudoscalar theory.

(e) Write the full relativisitc scattering amplitude at leading order in $g_{\pi NN}$ for nucleon-nucleon scattering in the pseudoscalar theory. There are two contributing Feynman diagrams.

(f) Write the full relativistic scattering amplitude at leading order in $g_{\pi NN}$ for nucleon-nucleon scattering in the pseudovector theory. Show that this is identical to the result for the pseudoscalar theory.

(g) In classical physics, one would not expect a pointlike neutron to interact with electromagnetic fields. However, the expectation value of the electromagnetic current in the neutron state receives a contribution from 1-loop vertex corrections. There are actually 2 diagrams, one in which the photon couples to a proton, the other in which it couples to a π^- meson. Write the expressions for these diagrams in the pseudoscalar theory, before combining denominators or doing any integrals. For the coupling of the photon to the π^-, use the Feynman rule in Problem 1.

(h) Compute these diagrams in the pseudoscalar theory, up to the integral over Feynman parameters. Show that the results for the form factor F_2 are finite both in the ultraviolet and in the infrared.

(i) Show explicitly that, when the diagrams are summed, $F_1(0) = 0$, as required if the neutron is to have zero electric charge.

(j) Compute the same diagrams to find the vertex correction in the pseudovector theory at the level of the calculation in part (g).

(k) Apply the simplification used earlier, and subtract the pseudoscalar result from the pseudovector result. You will find that some terms are left over, and that these are quadratically divergent! Also, you should find that the value of $F_1(0)$ in the pseudovector theory is not zero. Some additional diagrams must be added to give the full result. (Please see the solution set if you are curious about this.) Adding these diagrams cancels the quadratic divergence and sets $F_1(0) = 0$, but a logarithmic divergence in the order q^2 term remains.

This calculation fills in some details for a story about Richard Feynman that Feynman told in his Nobel Prize lecture and expanded upon in the (somewhat hagiographic) biography by Jagdish Mehra, "The Beat of the Different Drum" (Oxford University Press, 1994). It is one of my favorite Feynman stories, not least because it probably happened more or less as told. By 1949, theorists working in Quantum Electrodynamics knew about Feynman diagrams, but the word had not yet reached people working in nuclear physics, who were still using methods from the 1930's that were more cumberson and also not manifestly relativistically covariant. Here is the story as told in Mehra's book.

At the 1949 APS meeting, Murray Slotnik presented new results, which he had obtained after two years of calculation, concerning the interaction between an electron and a nuetron. He had found that the answers for the pseudoscalar theory and the pesudovector theory were different. In fact, in the case of the pseudovector theory, the answer was logarithmically divergent, but, in the pseudoscalar theory, it was convergent and gave well-defined answers. [J. Robert] Oppenheimer, who was in the audience, asked Slotnik: 'Well, what about Case's theorem?' By this he meant the new result of Kenneth Case (who, at that time, was a postdoctoral fellow at Institute for Advanced Study in Princeton), which was going to be reported the next day. [Oppenheimer was, at that time, the Director of the Institute.] Case had announced that he had proved that the results for both the pseudoscalar and pseudovector theories were the same. Slotnick answered: 'I never heard of Case's theorem!'

Feynman had missed Slotnick's talk, but somebody asked him about the discussion between Oppenheimer and Slotnick. He went to Slotnick and said: 'Look, I am very anxious to try out if I understand what these things mean. So just tell me what you did.' He replied, 'I scattered the electron off the neutron and I have a correction due to the mesons.'

Feynman described what happened next as follows: 'This was a welcome opportunity to test my guesses as to whether I really understood what these two couplings were. So I went home, and during the evening I worked out the electron-neutron scattering for the pseudoscalar and pseudovector couplings, saw that they were not equal and subtracted them, and worked out the difference in detail. The next day, at the meeting, I saw Slotnick and said, "Slotnick, I worked it out last night, I wanted to see if I got the same answers you do. I got a different answer for each coupling—but I would like to check in detail with you because I want to be sure of my methods." And he said, "What do you mean you worked it out last night? It took me six months!" And, when we compared the answers, he looked at mine and he asked, "What is that Q in there, that variable Q?" ... I said, "That's the momentum transferred by the electron, the electron deflected by different angles." "Oh," he said, "No, I only have the limiting value as Q approaches [0 for] the forward scattering." Well, it was easy enough to just substitute Q equals zero in my answers as he did. But it took him six months to do the case of zero momentum transfer, whereas, during one evening, I had done the finite and arbitrary momentum transfer. That was a thrilling moment for me, like receiving the Nobel Prize, because that convinced me, at last, I did have some kind of method and technique and understood how to do something that other people did not know how to do. That was my moment of triumph in which I realized I really had succeeeded in working out something worthwhile.' ...

As Feynman further recalled, next day Case reported his theorem at the APS meeting. 'And, just to be annoying, when Case finished, I said, "Yeah, but what about Slotnick's calculation?" You know, I mean Oppenheimer was imperious. If Case proved the theorem, it must be true. And I argued, "What about Slotnick's calculation? That theorem can't be true." And everybody laughed becuase it was perfectly logical to suppose the theorem is at fault rather than the calculation, you know. So [Oppenheimer] said: "Well, maybe Slotnick's calculation is wrong." I said, "No, I checked it last night and it's all right. I believe it's right." '