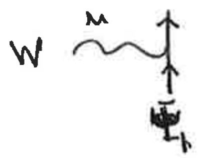
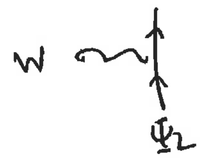
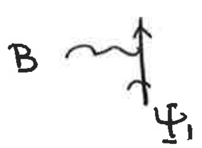
Physics 330 – Final Exam
Solutions

1.) a.) We first need to compute the fermion mass shift for a general vector boson mass. The Feynman rules are


$= -i\frac{g}{2}\gamma^\mu$


$= +ig\gamma^\mu$


$= -i\frac{g'}{2}\gamma^\mu$


$= -ig\gamma^\mu$

Let's work through the case of the shift of the Ψ_1 mass due to the W. In class, I did a similar calculation using Pauli Villars regularization, so here I'll do this with dimensional regularization

$-i\Sigma =$ 

$$= \left(-i\frac{g}{2}\right)^2 \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu\, i(\not{k}+M)\gamma_\mu}{k^2-M^2} \frac{-i}{(p-k)^2-m_W^2}$$

$$= -\frac{g^2}{4}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(-2\not{k}+4M)}{\left[x[(p-k)^2-m_W^2] + (1-x)(k^2-M^2)\right]^2}$$

$$= -\frac{g^2}{4}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{4M-2\not{k}}{\left[k^2-2x\,p\cdot k + x p^2 - x m_W^2 - (1-x)M^2\right]^2}$$

The denominator is $\quad \mathbb{k} = k - xp \quad k = \mathbb{k} + xp$

$$\mathbb{k}^2 + xp^2 - x^2p^2 - xm_W^2 - (1-x)M^2$$

$$= \mathbb{k}^2 - \Delta + i\varepsilon \qquad \Delta = (1-x)M^2 + xm_W^2 - x(1-x)p^2$$

$$-i\Sigma = -\frac{g^2}{4}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{4M - 2\not{\mathbb{k}} - 2x\not{p}}{[\mathbb{k}^2 - \Delta]^2}$$

$$= -\frac{g^2}{4}\int_0^1 dx \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)\ \Delta^{2-d/2}} (4M - 2x\not{p})$$

take the limit $d \to 4$ or $d = 4 - 2\epsilon$ with $\epsilon \to 0$

$$= -\frac{g^2}{4}\frac{i}{(4\pi)^2}\int_0^1 dx\,(4M - 2x\not{p})$$

$$\cdot\Big[\underbrace{\Big(\frac{1}{\epsilon} - \gamma + \log 4\pi\Big)}_{\text{call this } \log\Lambda^2} - \log\big[(1-x)M^2 + xm_W^2 - x(1-x)p^2\big]\Big]$$

this is the divergent term.

$$-i\Sigma(\not{p}) = -\frac{g^2}{4}\frac{i}{(4\pi)^2}\int_0^1 dx\,(4M - 2x\not{p})\log\frac{\Lambda^2}{[(1-x)M^2 + xm_W^2 - x(1-x)p^2]}$$

the mass shift, to $O(g^2)$ is

$$\Sigma(M) = \frac{g^2}{4(4\pi)^2}\int_0^1 dx\,(4-2x)M\,\log\Big[\frac{\Lambda^2}{(1-x)^2M^2 + xm_W^2}\Big]$$

call this $= \dfrac{g^2}{4}\mathcal{M}(M, m_W^2)$

Notice that $$\mathcal{M}(M,m) = \frac{1}{(4\pi)^2}\cdot 3M \cdot \log(\Lambda^2/M^2) + (\text{finite})$$

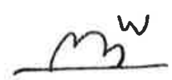
then the full mass shift is

for Ψ_1

$-i\Sigma_1 =$ 

$+$ 

$$\Sigma_1 = \frac{g^2}{4}\,\mathcal{M}(M,m_W) + \frac{(g')^2}{4}\,\mathcal{M}(M,m_B)$$

for Ψ_2 $-i\Sigma_1 =$ 

$+$ 

$$\Sigma_2 = \frac{(-g)^2}{4}\,\mathcal{M}(M,m_W) + \frac{(g')^2}{4}(M,m_B)$$

so

$\Sigma_1 = \Sigma_2$ up to the $\log\Lambda^2$ terms,

$$\Delta M = \frac{3(g^2+g'^2)}{64\pi^2}\, M \log(\Lambda^2/M^2)$$

b.) Try $A = \begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix}$ for the coefficients of $\begin{pmatrix}W\\ B\end{pmatrix}$

$$m^2\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix} = \frac{v^2}{4}\begin{pmatrix} g^2\sin\theta - gg'\cos\theta \\ -gg'\sin\theta + g'^2\cos\theta\end{pmatrix}$$

this gives 0 if $g\sin\theta = g'\cos\theta$

or $\sin\theta = \frac{g'}{\sqrt{g^2+g'^2}}$ $\cos\theta = \frac{g}{\sqrt{g^2+g'^2}}$

the orthogonal eigenvector ought to be

$$Z = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$$

Try it:

$$m^2 \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \frac{v^2}{4} \begin{bmatrix} g^2\cos\theta + gg'\sin\theta \\ -gg'\cos\theta - (g')^2\sin\theta \end{bmatrix}$$

plug in the above expressions for $\sin\theta$ and $\cos\theta$

$$= \frac{v^2}{4} \begin{bmatrix} g^2 \cdot \frac{g}{\sqrt{\ }} + gg' \frac{g'}{\sqrt{\ }} \\ -gg' \cdot \frac{g}{\sqrt{\ }} - (g')^2 \frac{g'}{\sqrt{\ }} \end{bmatrix}$$

$$= \frac{v^2}{4}(g^2 + g'^2) \begin{bmatrix} g/\sqrt{g^2+g'^2} \\ -g'/\sqrt{g^2+g'^2} \end{bmatrix} = \frac{(g^2+g'^2)v^2}{4} \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

so the orthogonal eigenvector is a vector boson with mass

$$m_Z^2 = \frac{(g^2+(g')^2)\,v^2}{4}$$

c.) Since we have seen that

$$A = \cos\theta\, B + \sin\theta\, W$$
$$Z = -\sin\theta\, B + \cos\theta\, W$$

then

$$B = \cos\theta\, A - \sin\theta\, Z$$
$$W = \sin\theta\, A + \cos\theta\, Z$$

plugging these into Hint, we find

$$H_{int} = \int d^3x \left[\frac{g}{2}(\cos\theta\, Z_\mu + \sin\theta\, A_\mu)(\bar\Psi_1\gamma^\mu\Psi_1 - \bar\Psi_2\gamma^\mu\Psi_2) + \frac{g'}{2}(-\sin\theta\, Z_\mu + \cos\theta\, A_\mu)(\bar\Psi_1\gamma^\mu\Psi_1 + \bar\Psi_2\gamma^\mu\Psi_2)\right]$$

$$= \int d^3x \Big[A_\mu \left(\frac{g}{2}\sin\theta + \frac{g'}{2}\cos\theta\right)\bar\Psi_1\gamma^\mu\Psi_1 + A_\mu\left(-\frac{g}{2}\sin\theta + \frac{g'}{2}\cos\theta\right)\bar\Psi_2\gamma^\mu\Psi_2 + Z_\mu\left(\frac{g}{2}\cos\theta - \frac{g'}{2}\sin\theta\right)\bar\Psi_1\gamma^\mu\Psi_1 + Z_\mu\left(-\frac{g}{2}\cos\theta - \frac{g'}{2}\sin\theta\right)\bar\Psi_2\gamma^\mu\Psi_2 \Big]$$

Notice that $\left(-\frac{g}{2}\sin\theta + \frac{g'}{2}\cos\theta\right) = \left(-\frac{gg'}{2} + \frac{g'g}{2}\right)\frac{1}{\sqrt{g^2+g'^2}} = 0$

also $\left(\frac{g}{2}\sin\theta + \frac{g'}{2}\cos\theta\right) = +\frac{gg'}{2}\times 2\,\frac{1}{\sqrt{g^2+g'^2}} = \frac{gg'}{\sqrt{g^2+g'^2}}$

so the electric charge of the Ψ_1 is $\frac{gg'}{\sqrt{g^2+g'^2}}$, which we can set equal to e

for Z the couplings are

Ψ_1: $\frac{g}{2}\cos\theta - \frac{g'}{2}\sin\theta = \frac{g^2-g'^2}{2}\frac{1}{\sqrt{g^2+g'^2}}$

Ψ_2: $-\frac{g}{2}\cos\theta - \frac{g'}{2}\sin\theta = -\left(\frac{g^2+g'^2}{2\sqrt{g^2+g'^2}}\right)$

in all, H_{int} becomes:

$$\mathcal{H}_{int} = \int d^3x \Big[\frac{gg'}{\sqrt{g^2+g'^2}} A_\mu \bar{\Psi}_1 \gamma^\mu \Psi_1 + 0$$
$$+ \frac{1}{2}\frac{(g^2-g'^2)}{\sqrt{g^2+g'^2}} Z_\mu \bar{\Psi}_1\gamma^\mu\Psi_1 - \frac{1}{2}\frac{g^2+g'^2}{\sqrt{g^2+g'^2}} Z_\mu \bar{\Psi}_2\gamma^\mu\Psi_2 \Big]$$

and the nonzero Feynman rules are

A  Ψ_1 $= -i\frac{gg'}{\sqrt{g^2+g'^2}}\gamma^\mu$

Z  Ψ_1 $= -i\frac{1}{2}\frac{g^2-g'^2}{\sqrt{g^2+g'^2}}\gamma^\mu$ Z  Ψ_2 $= +i\frac{1}{2}\frac{g^2+g'^2}{\sqrt{g^2+g'^2}}\gamma^\mu$

d.) We computed the general form of ΔM in part (a). Now we just need to evaluate this expression with the correct coupling constants and masses.

for Ψ_1 $\quad -i\Sigma =$  + 

$$\Delta M_1 = \frac{1}{4}\left(\frac{g^2-g'^2}{\sqrt{g^2+g'^2}}\right)^2 \mathcal{M}(M, m_Z^2) + \left(\frac{gg'}{\sqrt{g^2+g'^2}}\right)^2 \mathcal{M}(M,0)$$

for Ψ_2 $\quad -i\Sigma =$  (only)

$$\Delta M_2 = \frac{1}{4}\left(\frac{g^2+g'^2}{\sqrt{g^2+g'^2}}\right)^2 \mathcal{M}(M, m_Z^2)$$

To begin, pick out the $\log\Lambda^2$ term in each expression:

$$\mathcal{M}(M,m^2) \to \frac{3M}{(4\pi)^2}\log\Lambda^2$$

then

$$\Delta M_1 \Rightarrow \frac{1}{4(g^2+g'^2)}\left[(g^2-g'^2)^2+4(gg')^2\right]\cdot\frac{3M}{(4\pi)^2}\log\Lambda^2$$

$$\Delta M_2 \Rightarrow \frac{1}{4(g^2+g'^2)}\left[(g^2+g'^2)^2\right]\frac{3M}{(4\pi)^2}\log\Lambda^2$$

These are equal, so ΔM_1 and ΔM_2 are separately log-divergent, but $\Delta M_1 - \Delta M_2$ is finite and a prediction of the theory.

e.) Now we need to take account of the finite terms in ΔM from p. 2

$$\Delta M = \frac{3M}{(4\pi)^2}\log\Lambda^2/M^2 + \frac{M}{(4\pi)^2}\int_0^1 dx\,(4-2x)\log\left[\frac{M^2}{(1-x)^2M^2+xm^2}\right]$$

You need to be a little careful with this integral. Naively

$$\int_0^1 dx\,\log\left((1-x)^2M^2+xm^2\right)$$

$$= \int_0^1 dx\left[\log(1-x)^2M^2 + \frac{xm^2}{(1-x)^2M^2}+\cdots\right]$$

but this integral is very divergent as $x\to 1$ so this expansion is not valid.

The difficulty is only for x near 1, so we can set $x=1$ in the xm^2 term. Now try to treat exactly

$$\int_0^1 dx\,\log\left((1-x)^2M^2+m^2\right)$$

write $(1-x)^2 M + m^2$

$$= y^2 M^2 + m^2 \qquad y = (1-x)$$

the ~~integral~~ becomes

$$\int_0^1 dy \ \left[\log (yM + im) + \log (yM - im) \right]$$

$$= \frac{1}{M} \Big[(yM+im) \lg (yM+im) - (yM+im)$$
$$+ (yM-im) \lg (yM-im) - (yM-im) \Big] \Big|_0^1$$

$$= \frac{1}{M} \Big[(M+im) \lg (M+im) + (M-im) \lg (M-im) - 2M$$
$$- im \lg (im) - (-im) \lg (-im) \Big]$$

$$= \frac{1}{M} \Big[(M+im)\left(\lg M + i \frac{m}{M} + \cdots\right) + (M-im)\left(\lg M - \frac{im}{M} + \cdot\right)$$
$$-2M$$
$$- im\left(\lg (m) + i \frac{\pi}{2}\right) + im \lg\left(m - i\frac{\pi}{2}\right) \Big] + O\left(\frac{m^2}{M^2}\right)$$

$$= \frac{1}{M} \Big[2M \lg M - 2M + \underline{2m} \cdot \frac{\pi}{2} + O\left(\frac{m^2}{M}\right)$$

agin

$$\int_0^1 dx \ \log \left((1-x)^2 M^2 + m^2 \right)$$

$$= 2 \lg M - 2 + \frac{m}{M} \cdot \pi + O\left(\frac{m^2}{M^2}\right)$$

We do need to think a little more about the full integral

$$\int_0^1 dx \ (4-2x) \log \left((1-x)^2 M + x m^2 \right)$$

write this as

$$\int_0^1 dy\ (2+2y)\log\left(y^2M^2 + (1-y)m^2\right)$$

At $m^2=0$ this is

$$\int_0^1 dy\ (2+2y)\cdot 2\left(\log M + \log y\right)$$

$$= 2\cdot 3\log M + 2\cdot 2\underbrace{\int_0^1 dy \log y}_{-1} + 2\cdot 2\underbrace{\int_0^1 dy\, y\log y}_{-\frac{1}{4}}$$

$$= 3\log M^2 - 5$$

If we expand $\log[y^2M^2 + (1-y)m^2] \cong \log y^2M^2 + \frac{(1-y)}{y^2}\frac{m^2}{M^2}$

near $y=0$

$$y\log[y^2M^2 + (1-y)m^2] \cong y\log y^2M^2 + \frac{y}{y^2}\frac{m^2}{M^2}$$

the correction terms will be $\sim \frac{m^2}{M^2}\log M^2/m^2$. (You can check this more rigorously)

so

$$\int_0^1 dx\ (4-2x)\log\left((1-x)^2M + xm^2\right)$$

$$= 3\log M^2 - 5 + 2\cdot\frac{\pi m}{M} + O\left(m^2/M^2\right)$$

then we can assemble

(from p. 6)

$$\Delta M_1 = \frac{(g^2+g'^2)^2}{4(g^2+g'^2)} \frac{3M}{(4\pi)^2} \log \Lambda^2/M^2$$

$$+ \frac{(g^2-g'^2)^2}{4(g^2+g'^2)} \frac{M}{16\pi^2} (-1) \left[-5 + \frac{2\pi m_Z}{M} + O(m^2/M^2) \right]$$

$$+ \frac{4(gg')^2}{4(g^2+g'^2)} \frac{M}{16\pi^2} (-1) \left[-5 + (0) + O(m_Z^2/M^2) \right]$$

$$\Delta M_2 = \frac{(g^2+g'^2)^2}{4(g^2+g'^2)} \frac{M}{16\pi^2} (-1) \left[-5 + \frac{2\pi m_Z}{M} + O(m_Z^2/M^2) \right]$$

$$\Delta M_1 - \Delta M_2 = + \frac{4 g^2 g'^2}{4(g^2+g')} \frac{M}{16\pi^2} \cdot \frac{2\pi m_Z}{M} + O(m_Z^2/M^2)$$

$$= \frac{g^2 g'^2}{g^2+g'^2} \frac{1}{16\pi^2} 2\pi m_Z$$

$$= \left(\frac{\alpha_w \alpha'}{\alpha_w + \alpha'} \right) \frac{1}{4\pi} 2\pi m_Z$$

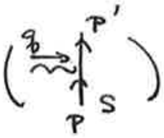
$$\Delta M_1 - \Delta M_2 = \frac{1}{2} \left(\frac{\alpha_w \alpha'}{\alpha_w + \alpha'} \right) m_Z$$

f) $m_Z^2 = \frac{g^2+(g')^2}{4} v^2 = \pi(\alpha_w + \alpha') v^2 = (91 \text{ GeV})^2$

$$\Delta M_1 - \Delta M_2 = 350 \text{ MeV}$$

[for the original insight, see
S. Thomas and J. Wells, PRL 81 34 (1997), arXiv: hep-ph/9609434]

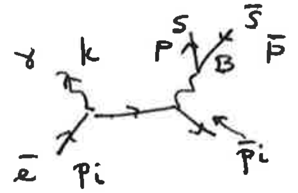
2.) a.)

$$q^\mu \cdot \left(\right.$$  $$\left. \right) = q^\mu (-ig)(p+p')_\mu$$

$$q = p' - p \qquad\qquad = -ig\,(p'^2 - p^2)$$

If the scalars are on shell $p^2 = p'^2 = M^2$ so this $= 0$

b) $e^+e^- \to S\bar{S} + \gamma$



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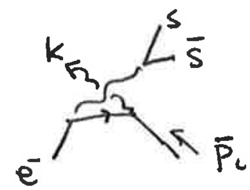
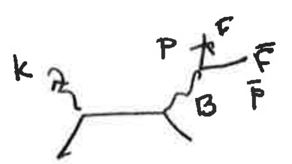


$e^+e^- \to F\bar{F} + \gamma$



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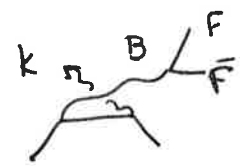


c.) For $e^+e^- \to S\bar{S}\gamma$

$$i\mathcal{M} = (-ie)(-ig)^2\ \bar{v}(\bar{p}_i)\left[\gamma^\mu\left(\frac{i(\not{p}_i - \not{k})}{(p_i-k)^2}\right)\gamma^\nu + \gamma^\nu\frac{i(\not{k}-\bar{\not{p}}_i)}{(k-\bar{p}_i)^2}\gamma^\mu\right]u(p_i)$$

$$\cdot\ (p+\bar{p})_\mu \cdot \varepsilon^*_\nu(k)\cdot\left(\frac{-i}{(p+\bar{p})^2 - m_B^2}\right)$$

$$i\mathcal{M} = -ieg^2\ \bar{v}(\bar{p}_i)\left[\gamma^\mu\left(\frac{\not{p}_i - \not{k}}{-2p_i k}\right)\gamma^\nu + \gamma^\nu\left(\frac{\not{k}-\bar{\not{p}}_i}{-2\bar{p}_i\cdot k}\right)\gamma^\mu\right]u(p_i)$$

$$\cdot\ (p+\bar{p})_\mu\ \varepsilon^*_\nu(k)\ \frac{1}{(p+p'-m_B^2)}$$

similarly for $e^+e^- \to F\bar{F}\gamma$

$$i\mathcal{M} = -ieg^2\, \bar{v}(\bar{p}_i)\left[\gamma^\mu \frac{(\not{p}_i-\not{k})}{-2p_i k}\gamma^\nu + \gamma^\nu\left(\frac{\not{k}-\bar{\not{p}}_i}{-2\bar{p}_i k}\right)\gamma^\mu\right]u(p_i)$$

$$\cdot\ \bar{u}(p)\gamma_\mu v(\bar{p})\ \varepsilon_\nu^*(k)\ \frac{1}{(p+p')^2-m_B^2}$$

d.) If $M_B \gg E_{cm}$

$$\frac{g^2}{(p+p')^2-m_B^2} \approx -\frac{g^2}{m_B^2}$$

so the cross section depends only on this combination

e.) $$\int d\Pi_3 = \int \frac{d^3k\, d^3p\, d^3\bar{p}}{(2\pi)^3 2k\,(2\pi)^3 2E_p (2\pi)^3 2E_{\bar{p}}}(2\pi)^4 \delta^{(4)}(p_i+\bar{p}_i-(k+p+\bar{p}))$$

insert $$1 = \int \frac{d^4P}{(2\pi)^4}(2\pi)^4\delta^{(4)}(P-(p+\bar{p}))$$

$$\int d\Pi_3 = \int\frac{d^3k}{(2\pi)^3}\frac{1}{2k}\int\frac{d^4P}{(2\pi)^4}(2\pi)^4\delta(p_i+\bar{p}_i-(k+P))$$

$$\cdot \int\frac{d^3p\, d^3\bar{p}}{(2\pi)^3 2E_p(2\pi)^3 2E_{\bar{p}}}(2\pi)^4\delta^{(4)}(P-(p+\bar{p}))$$

$$= \int\frac{d^3k}{(2\pi)^3}\frac{1}{2k}\int\frac{d^4P}{(2\pi)^4}(2\pi)^4\delta(p_i+\bar{p}_i-(k+P))\cdot\int d\Pi_2(p,\bar{p})$$

To change $\int d^4P$ into $\int d^3P$, add

$$1 = \int\frac{dm_P^2}{(2\pi)}2\pi\,\delta(m_P^2-P^2)$$

$$\int\frac{dP^0}{2\pi}2\pi\,\delta(m_P^2-P^2) = \frac{1}{2E_P} \quad \text{with } E_P = \left[m_P^2+(\vec{P})^2\right]^{\frac{1}{2}}$$

then

$$\int d\Pi_3 = \int \frac{dm_P^2}{2\pi} \int \frac{d^3k}{(2\pi)^3 2k} \int \frac{d^3P}{(2\pi)^3 2E_P} (2\pi)^4 \delta(p_i + \bar{p}_i - (k+P))$$

$$\cdot \int d\Pi_2(p, \bar{p})$$

$$= \int \frac{dm_P^2}{2\pi} \int d\Pi_2(k, P) \int d\Pi_2(p, \bar{p})$$

Remember that $\int d\Pi_2$ is invariant; we can evaluate it in any frame.

f.) In the rest frame of P^μ, $P^\mu = (m_P, \vec{0})$ where $m_P^2 = P^2$. Then in this frame

$$\frac{P^\mu}{\sqrt{P^2}} = \frac{(m_P, \vec{0})^\mu}{m_P} = (1, \vec{0})^\mu$$

$\frac{P^\mu}{\sqrt{P^2}}$ is a covariant expression that agrees with this in 1 frame

so this represents the vector in any frame

Similarly, in this frame,

$$-\left(\eta^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

so $\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ in this frame $= -\left(\eta^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right)$

in any frame

g) Following the steps as suggested in the problem set:

(1) the trace over the final fermions is

$$\mathrm{tr}\left[\gamma^\mu (\not{p} + m)\gamma^\lambda (\bar{\not{p}} - M)\right]$$

$$= 4\left[p^\mu \bar{p}^\lambda + p^\lambda \bar{p}^\mu - \eta^{\mu\nu}\, p\cdot\bar{p} - \eta^{\mu\nu} M^2\right]$$

Integrate this over the $(p, \bar{p})$ phase space, work in the rest frame of $P = (p + \bar{p})$. In this frame

$$p = (E_p, p\hat{n}) \qquad \bar{p} = (E_p, -p\hat{n}) \qquad E_p = [p^2 + M^2]^{\frac{1}{2}}$$

$$P^0 = m_P = 2E_p$$

In this frame, the trace is

$$4\left[\left(\begin{array}{c|c} E_p^2 & -E_p n^j \\ \hline E_p p n^i & -p^2 n^i n^j \end{array}\right) + \begin{pmatrix} E_p^2 & +E_p p n^j \\ -E_p p n^i & -p^2 n^i n^j \end{pmatrix} + \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\cdot(\underbrace{E_p^2 + p^2 + M^2}_{= 2E_p^2})\right]$$

$$= 4\left[\begin{array}{cc} 0 & 0 \\ 0 & (2E_p^2\,\delta^{ij} - 2p^2 n^i n^j) \end{array}\right]$$

(2) Average over the direction of $\hat{n}$:

$$= 4\left[\begin{array}{cc} 0 & 0 \\ 0 & \delta^{ij}\,(2E_p^2 - \frac{2}{3}p^2) \end{array}\right]$$

$$= 4\begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\left(\frac{4}{3}E_p^2 + \frac{2}{3}M^2\right)$$

$$= 4 \cdot \left(\frac{P^2}{3} + \frac{2M^2}{3} \right) \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}^{\mu\lambda}$$

The phase space integral is then

$$\frac{1}{8\pi} \sqrt{1 - \frac{4M^2}{P^2}} \cdot \frac{4}{3}\left(P^2 + 2M^2\right) \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}^{\mu\lambda}$$

[This same factor

$$\frac{4}{3} \left(s + 2M^2\right)\left(1 - \frac{4M^2}{s}\right)^{\frac{1}{2}}$$

appears in the $e^+e^- \to F\bar{F}$ total cross section.]

(3) $$= \frac{1}{8\pi} \sqrt{1 - \frac{4M^2}{P^2}} \; \frac{4}{3} \left(P^2 + 2M^2\right) \left(-\eta^{\mu\lambda} + \frac{P^\mu P^\nu}{P^2} \right)$$

(4) Now we need to do the electron trace. But first note that, by the Ward-Takahashi identity

$$P_\mu \cdot \bar{v}(\bar{p}_i) \left[\gamma^\mu \frac{(\not{p}_i - \not{k})}{(p_i - k)^2} \gamma^\nu + \gamma^\nu \frac{\not{k} - \bar{\not{p}}_i}{(k - \bar{p}_i)^2} \right] u(p_i)$$

$$= 0$$

so we can reduce the tensor in (3) to $-\eta^{\mu\lambda}$

Then we have to evaluate:

$$\mathrm{tr} \Big[\bar{\not{p}}_i \Big[\gamma^\mu \frac{(\not{p}_i - \not{k})}{-2p_i k} \gamma^\nu + \gamma^\nu \frac{(\not{k} - \bar{\not{p}}_i)}{-2k \cdot \bar{p}_i} \gamma^\mu \Big] \not{p}_i$$

$$\Big[\gamma^\sigma \frac{\not{p}_i - \not{k}}{-2p_i \cdot k} \gamma^\lambda + \gamma^\lambda \frac{(\not{k} - \bar{\not{p}}_i)}{-2k \cdot \bar{p}_i} \gamma^\sigma \Big] \Big]$$

$$\cdot \left(- \eta^{\mu\lambda}\right) \left(-\eta^{\nu\sigma}\right)$$

trace over photon polarizations

$$= \frac{\mathrm{I}}{(2p_i\cdot k)^2} + \frac{\mathrm{II}}{2p_i\cdot k\, 2\bar{p}_i\cdot k} + \frac{\mathrm{III}}{2\bar{p}_i k\; 2p\, k} + \frac{\mathrm{IV}}{(2\bar{p}_i\cdot k)^2}$$

with

$$\mathrm{I} = tr\,[\bar{\not p}_i \gamma^\mu (\not p_i - \not k)\gamma^\nu \not p_i \gamma_\nu (\not p_i - \not k)\gamma_\mu]$$

$$= (-2)^2\, tr\,[\bar{\not p}_i (\not p_i - \not k) \not p_i (\not p_i - \not k)]$$

$$= 4\cdot 4\cdot [\,2\bar{p}_i\cdot(p_i-k)\; p_i(p_i-k) - \bar{p}_i\cdot p_i\,(p_i-k)^2\,]$$

$$\mathrm{II} = tr\; \bar{\not p}_i \gamma^\nu (\not p_i - \not k)\gamma^\nu \not p_i \gamma_\mu (k-\bar{p}_i)\gamma_\nu$$

$$= (-2)\; tr\; \bar{\not p}_i \gamma^\mu (p_i-k)(k-\bar{p}_i)\gamma_\mu \not p_i$$

$$= (-2)(+4)(p_i\bar{p}_i)\; tr\,(p_i-k)(k-\bar{p}_i)$$

$$= -8\cdot 4\;\; p_i\bar{p}_i\;\; (p_i-k)(k-\bar{p}_i)$$

$$\mathrm{III} = tr\; \bar{p}_i \gamma^\nu (k-\bar{p}_i)\gamma^\mu \not p_i \gamma_\nu (p_i-k)\gamma_\mu$$

$$= -2\; tr\; \bar{p}_i \gamma^\nu (k-\bar{p}_i)(p_i-k)\gamma_\nu \not p_i$$

$$= (-2)(+4)\;\bar{p}_i p_i\; tr\,(k-\bar{p}_i)(p_i-k)$$

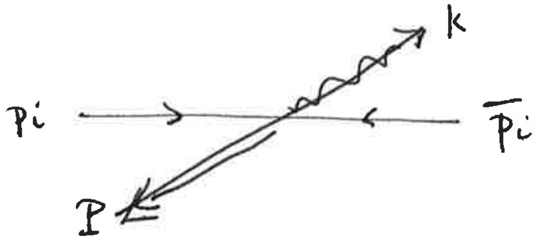
$$= -8\cdot 4\;\; p_i\bar{p}_i\;\; (p_i-k)\cdot(k-\bar{p}_i)$$

$$\mathrm{IV} = tr\; \bar{\not p}_i \gamma^\nu (\not k - \bar{\not p}_i)\gamma^\mu \not p_i \gamma_\mu (k-\bar{p}_i)\gamma_\nu$$

$$= (-2)^2\, tr\; \bar{\not p}_i (k-\bar{p}_i) \not p_i\, (k-\bar{p}_i)$$

$$= 4\cdot 4\cdot [\,2\bar{p}_i\cdot(k-\bar{p}_i)\; p_i\cdot(k-\bar{p}_i) - p_i\bar{p}_i\,(k-\bar{p}_i)^2\,]$$

We need to do some kinematics to evaluate the invariants



$p_i = (E, 0, 0, E)$ $\qquad$ $\bar{p}_i = (E, 0, 0, -E)$

$k = (k,\ k\sin\theta, 0\ k\cos\theta)$ $\qquad$ $P = (E_P, -k\sin\theta, 0, -k\cos\theta)$

$$k + E_p = 2E$$
$$k + [m_P^2 + k^2]^{1/2} = 2E$$
$$(m_P^2 + \cancel{k^2}) = 4E^2 - 4Ek + \cancel{k^2}$$
$$\text{so} \quad k = \frac{4E^2 - m_P^2}{4E} = E - \frac{m_P^2}{4E}$$

Obviously, we need $E > m_P^2 > (2M)^2$ to produce dark matter

then

$$p_i \cdot \bar{p}_i = 2E^2 \qquad p_i^2 = \bar{p}_i^2 = 0 = k^2$$
$$p_i \cdot k = kE(1-\cos\theta) \qquad \bar{p}_i \cdot k = kE(1+\cos\theta)$$
$$(p_i - k)^2 = -2p_i k = -2kE(1-\cos\theta) \quad (k-\bar{p})^2 = -2Ek(1+\cos\theta)$$

then

$$\text{(I)} = 16\left[2(2E^2 - kE(1+\cos\theta))(-Ek(1-\cos\theta)) - 2E^2(-2kE(1-\cos\theta))\right]$$
$$= 32\,Ek(1-\cos\theta)\left[-2E^2 + Ek(1+\cos\theta) + 2E^2\right]$$
$$= 32\,(Ek)^2(1-\cos\theta)(1+\cos\theta)$$
$$\text{(IV)} = 16\left[2kE(1+\cos\theta)(Ek(1-\cos\theta) - 2E^2) - 2E^2(-2Ek(1+\cos\theta))\right]$$
$$= 32\,(Ek)^2(1+\cos\theta)(1-\cos\theta)$$
$$\text{(II)} = \text{(III)} = -32(2E^2)\left[kE(1-\cos\theta) + kE(1+\cos\theta) - 2E^2\right]$$
$$= -64\,E^2\,E\,2(E-k) = -32 \cdot 4 \cdot E^3(E-k)$$

in all $\mathrm{tr}[\cdot\cdot]$

$$= \frac{32}{4}\frac{1}{(kE)^2}\left\{ \frac{(kE)^2(1-\cos\theta)(1+\cos\theta)}{(1-\cos\theta)^2} + 2\cdot\frac{4E^3(E-k)}{(1+\cos\theta)(1-\cos\theta)} + \frac{(kE)^2(1-\cos\theta)(1+\cos\theta)}{(1+\cos\theta)^2}\right\}$$

$$= 8\left\{ \frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta} + \frac{8E^3(E-k)/(Ek)^2}{(1+\cos\theta)(1-\cos\theta)}\right\}$$

$$= \frac{8}{1-\cos^2\theta}\left\{ 2+2\cos^2\theta + 8E(E-k)/k^2\right\}$$

$$E-k = \frac{m_P^2}{4E}$$

$$= \frac{8\cdot 2}{1-\cos^2\theta}\left\{ 1+\cos^2\theta + \frac{m_P^2}{k^2}\right\}$$

$$= \frac{16}{1-\cos^2\theta}\left\{ 1+\cos^2\theta + \frac{m_P^2}{k^2}\right\}$$

Basically, we have computed the cross section for $e^+e^- \to \gamma +$ (heavy vector boson)

That cross section would be

$$d\sigma = \frac{1}{2S}\frac{1}{8\pi}\int\frac{d\cos\theta}{2}\frac{k}{E}\cdot\frac{(eg)^2}{4}\sum|M|^2$$

$$= \frac{1}{32\pi E_{cm}^2}\, d\cos\theta\left(\frac{E_{cm}^2-m_P^2}{E_{cm}^2}\right) e^2g^2\, 4\left(\frac{1+\cos^2\theta + m_P^2/k^2}{1-\cos^2\theta}\right)$$

$S = E_{cm}^2$

$$= \frac{2\pi\alpha\,(g^2/4\pi)}{S}\, d\cos\theta\left(1-\frac{m_P^2}{S}\right)\left(\frac{1+\cos^2\theta + m_P^2/k^2}{1-\cos^2\theta}\right)$$

Now add the other elements of the cross section for $e^+e^- \to F\bar{F}\gamma$

$$d\sigma = \int \frac{dm_P^2}{2\pi} \frac{1}{m_B^2} \frac{2\pi\alpha}{s} \frac{g^2}{4\pi} \left(1 - \frac{m_P^2}{s}\right) d\cos\theta$$

$$\cdot \left(\frac{1+\cos^2\theta + m_P^2/k^2}{1-\cos^2\theta} + \frac{m_P^2/k^2}{1-\cos^2\theta} \right)$$

$$\cdot \frac{1}{8\pi} \sqrt{1 - \frac{4M^2}{m_P^2}} \; \frac{4}{3}\left(m_P^2 + 2M^2\right) \leftarrow$$ (from (3) on p.15)

We need to exchg k for $m_P^2 = P^2$ or vice versa. Probably it is easier to use m_P^2 except to note

$$k = E - \frac{m_P^2}{4E} \qquad \text{so} \qquad dm_P^2 = -4E\, dk$$

and $$K = E\left(1 - \frac{m_P^2}{s}\right) \quad \text{or} \quad m_P^2 =$$

then

$$d\sigma = dk\, d\cos\theta \; \frac{4E}{2\pi} \left(\frac{1}{m_B^2}\right)^2 \frac{2\pi\alpha}{s} \frac{(g^2)^2}{4\pi} \left(1 - \frac{m_P^2}{s}\right) \left(\frac{1}{1-\cos^2\theta}\right)$$

$$\cdot \left\{ 1 + \cos^2\theta + \frac{m_P^2}{E^2(1 - m_P^2/s)^2} \right\}$$

$$\cdot \frac{1}{8\pi} \sqrt{1 - \frac{4M^2}{m_P^2}} \cdot \frac{4}{3}\left(m_P^2 + 2M^2\right)$$

$$\frac{d\sigma}{dk\, d\cos\theta} = \frac{\alpha}{8\pi^2} \left(\frac{g^2}{m_B^2}\right)^2 \frac{E}{s} \frac{1}{1-\cos^2\theta}$$

$$\left\{ (1+\cos^2\theta)\left(1 - \frac{m_P^2}{s}\right) + \frac{4m_P^2}{s} \frac{1}{(1 - m_P^2/s)} \right\}$$

$$\cdot \frac{4}{3} \sqrt{1 - \frac{4M^2}{m_P^2}} \left(m_P^2 + 2M^2\right)$$

or, finally

$$\frac{d\sigma}{dk\, d\cos\theta} = \frac{\alpha}{8\pi^2}\left(\frac{g^2}{m_B^2}\right)^2 \frac{1}{\sqrt{s}} \frac{1}{1-\cos^2\theta}$$

$$\cdot \left\{ (1+\cos^2\theta)(1-m_P^2/s) + \frac{4m_P^2}{s}\frac{1}{(1-m_P^2/s)} \right\}$$

$$\cdot \left[\frac{4}{3}\sqrt{1-4M^2/m_P^2}\;(m_P^2+2M^2) \right]$$

h.) Now repeat this analysis for the reaction

$$e^+e^- \to S\bar{S}\gamma$$

Step (1) is to compute the integral over the $(p,\bar{p})$ phase space of

$$(p+\bar{p})^\mu(p+\bar{p})^\lambda$$

Looking back to p. 14 $(p+\bar{p})^\mu = (0,\, 2p\hat{n})$

so $\int d\Pi_2\, (p+\bar{p})^\mu (p+\bar{p})^\lambda$

$$= \int d\Pi_2 \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 4p^2 n^i n^j \end{array}\right) = \int d\Pi_2 \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 4p^2\cdot\frac{1}{3}\delta^{ij} \end{array}\right)$$

$$= \frac{\left(1-\frac{4M^2}{m_P^2}\right)^{1/2}}{8\pi}\frac{4}{3}p^2 \begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$$

$$= \frac{\left(1-\frac{4M^2}{m_P^2}\right)^{1/2}}{8\pi}\frac{1}{3}(m_P^2-4M^2)\begin{pmatrix} 0&&&\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}$$

$$= \frac{1}{8\pi}\frac{1}{3}m_P^2\left(1-\frac{4M^2}{m_P^2}\right)^{3/2}\left[-\left(\eta^{\mu\lambda}-\frac{P^\mu P^\lambda}{P^2}\right)\right]$$

the remaining steps are identical to those in part (g)

We find for this case

$$\frac{d\sigma}{dk\, d\cos\Theta} = \frac{\alpha}{8\pi^2}\left(\frac{g^2}{m_B^2}\right)^2 \frac{1}{\sqrt{s}}\left(\frac{1}{1-\cos^2\Theta}\right)$$

$$\left\{ (1+\cos^2\Theta)\left(1-\frac{m_P^2}{s}\right) + \frac{4m_P^2}{s}\frac{1}{1-m_P^2/s} \right\}$$

$$\cdot \left[\frac{1}{3}\left(1-\frac{4M^2}{m_P^2}\right)^{3/2} \cdot m_P^2 \right.$$

i.) To emphasize the difference between these results, I've plotted the dimensionless functions

$$FF: \quad \frac{1}{1-\cos^2\Theta}\left\{(1+\cos^2\Theta)\left(1-\frac{m_P^2}{s}\right) + \frac{4m_P^2}{s}\left(\frac{1}{1-m_P^2/s}\right)\right\} \frac{4}{3}\left(1-\frac{4M^2}{m_P^2}\right)^{1/2}\left(\frac{m_P^2+2M^2}{s}\right)$$

$$SS: \quad \frac{1}{1-\cos^2\Theta}\left\{(1+\cos^2\Theta)\left(1-\frac{m_P^2}{s}\right) + \frac{4m_P^2}{s}\left(\frac{1}{1-m_P^2/s}\right)\right\} \frac{4}{3}\left(1-\frac{4M^2}{m_P^2}\right)^{3/2} \frac{m_P^2}{s}$$

note, I changed this coefficient so that for $M^2 \ll m_P^2$ the normalizations are equal

Here are plots for

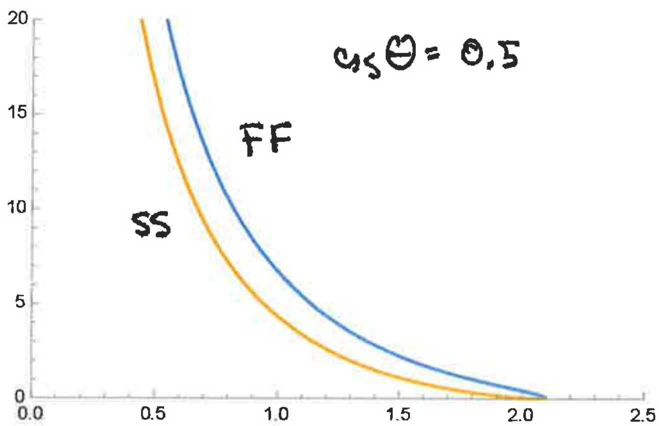
$M = 1 \quad E_{cm} = 5$

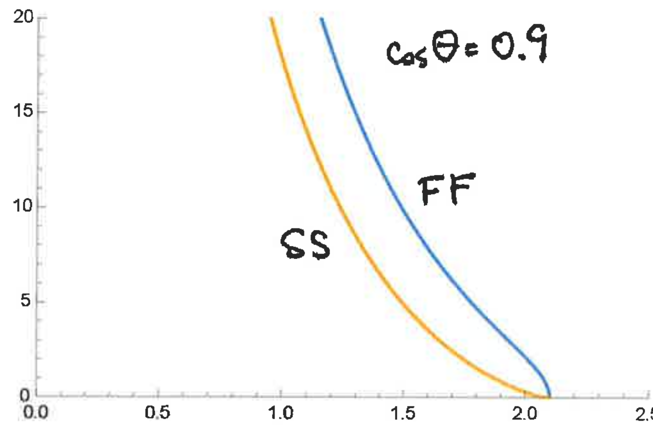
$M = 1 \quad E_{cm} = 3$

$$FF: \quad \frac{1}{1-\cos^2\Theta}\left\{(1+\cos^2\Theta)\left(1-\frac{m_P^2}{s}\right) + \frac{4m_P^2}{s}\frac{1}{1-m_P^2/s}\right\} \frac{4}{3}\left(1-\frac{4M^2}{m_P^2}\right)^{1/2}\left(\frac{m_P^2+2M^2}{s}\right)$$

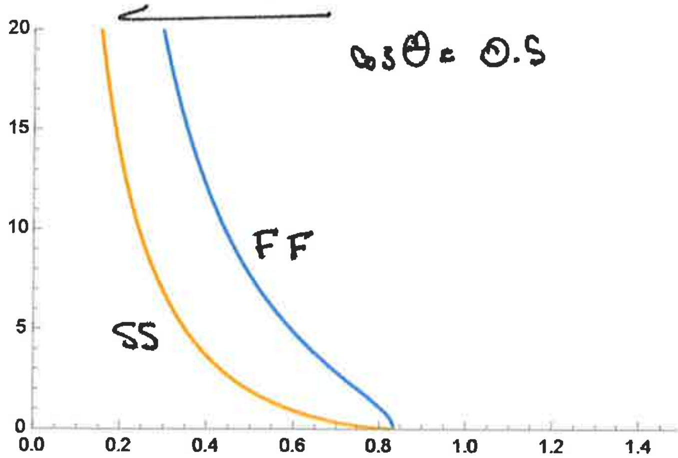
$$SS: \quad \frac{1}{1-\cos^2\Theta}\left\{(1+\cos^2\Theta)\left(1-\frac{m_P^2}{s}\right) + \frac{4m_P^2}{s}\frac{1}{1-m_P^2/s}\right\} \frac{4}{3}\left(1-\frac{4M^2}{m_P^2}\right)^{3/2} \frac{m_P^2}{s}$$

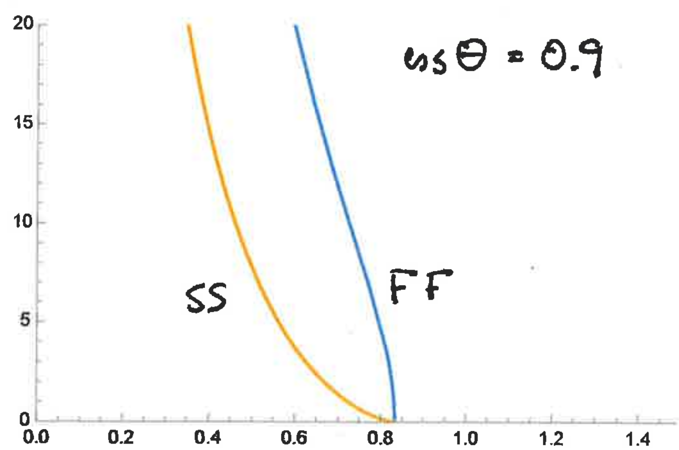
$M = 1 \quad E_{cm} = 5$





$M = 1 \quad E_{cm} = 3$





j.) The maximum photon energy in these reactions is given by the minimum value of m_{ℓ}^2, which is $(2M)^2$

Then this maximum value k_{max} is given by

$$k_{max} = \frac{s - (2M)^2}{2\sqrt{s}}$$

or

$$M = \frac{1}{2}\left[E_{cm}\left(E_{cm} - 2k_{max}\right)\right]^{1/2}$$

k.) Near k_{max}, the cross section behaves as

$$\frac{d\sigma}{dk} \sim \left[k_{max} - k\right]^{1/2} \quad \text{for } F\bar{F}$$

$$\sim \left[k_{max} - k\right]^{3/2} \quad \text{for } S\bar{S}$$

this behavior is apparent in the plots on p.22