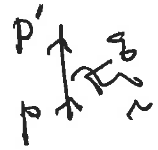
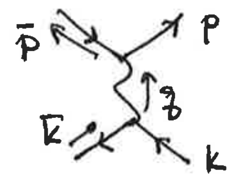
Physics 330 – Final Exam
Solutions

1.) a.) 

$$\cdot\, q^\mu = (-ie)(p'+p)\cdot(p'-p)$$

$$= (-ie)\left[(p')^2 - p^2\right] = 0$$

if $(p')^2 = p^2 = m^2$

b.) The Feynman diagram is 

$$i\mathcal{M} = (-ie)^2\, \bar{v}(\bar{k})\gamma^\mu u(k)\, \frac{-i}{q^2}\,(p-\bar{p})_\mu$$

The square of the electron part is

$$\sum_{\text{spins}} |\bar{v}(\bar{k})\gamma^\mu u(k)|^2 = \mathrm{tr}\left[\bar{k}\gamma^\mu k \gamma^\nu\right]$$

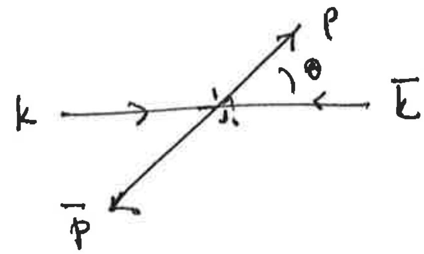
$$= 4\left[\bar{k}^\mu k^\nu + \bar{k}^\nu k^\mu - \bar{k}\cdot k\, g^{\mu\nu}\right]$$

Then, averaging over initial spins,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{4}\cdot 4e^4\left[\bar{k}^\mu k^\nu + \bar{k}^\nu k^\mu - k\cdot\bar{k}\, g^{\mu\nu}\right]\cdot\frac{1}{(q^2)^2}$$

$$\cdot\,(p-\bar{p})^\mu (p-\bar{p})^\nu$$

the kinematics in the CM frame is



$$k = (E, 0, 0, E) \qquad p = (E, ps, 0\ pc) \qquad (p-\bar{p}) = (0\ 2ps, 0, 2pc)$$

$$\bar{k} = (E, 0\ 0\ -E) \qquad \bar{p} = (E, -ps, 0, -pc) \qquad \begin{array}{l} s = \sin\theta \\ c = \cos\theta \end{array}$$

$$\text{with} \quad p = \left[E^2 - m^2\right]^{1/2} \qquad E_{cm} = 2E$$

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 = e^4\left[(+2Epc)(-2Epc)\cdot 2 - (2E^2)(-4p^2)\right]\frac{1}{(2E)^4}$$

$$= e^4 \cdot 8\,\frac{\left(E^2p^2 - E^2p^2\cos^2\theta\right)}{16E^4} = \frac{8e^4}{16E^2}\,p^2\sin^2\theta$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2E\cdot 2E\cdot 2}\,\underbrace{\frac{1}{8\pi}\cdot\frac{1}{2}\cdot\frac{p}{E}}_{\text{phase space}}\cdot\frac{2e^4}{E_{cm}^2}\,p^2\sin^2\theta$$

$$= \frac{e^4}{16\cdot 4\pi}\,\frac{1}{E_{cm}^2}\left(\frac{p}{E}\right)^3\sin^2\theta$$

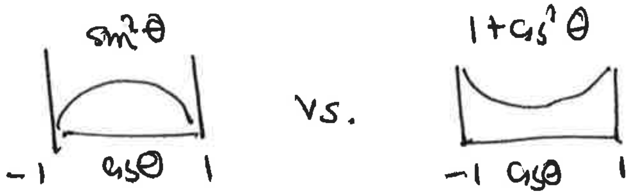
$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{4E_{cm}^2}\left(1 - 4m^2/E_{cm}^2\right)^{3/2}\sin^2\theta$$

$$\int_{-1}^{1} d\cos\theta\,\sin^2\theta = \int_{-1}^{1} dc\,(1-c^2) = 2 - 2/3 = \frac{4}{3}$$

$$\sigma_{tot.} = \frac{\pi\alpha^2}{3E_{cm}^2}\left(1 - 4m^2/E_{cm}^2\right)^{3/2}$$

comparison with $e^+e^- \to \mu^+\mu^-$

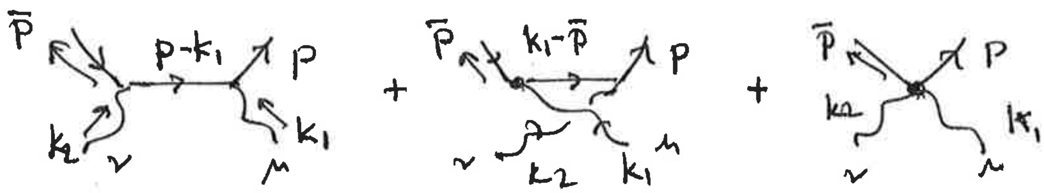
- total cross section is smaller by a factor 4
- angular distribution is opposite



- threshold behavior is much slower

$$\left(1-\frac{4m^2}{E_{cm}}\right)^{3/2} \quad \text{vs.} \quad \left(1+\frac{2m^2}{E_{cm}^2}\right)\left(1-\frac{4m^2}{E_{cm}^2}\right)^{1/2}$$

c.)



$$i\mathcal{M} = (-ie)^2 (\varepsilon_{1\mu}\varepsilon_{2\nu})$$

$$\cdot \Big[(2p-k_1)^\mu \frac{i}{(p-k_1)^2-m^2}(p-k_1-\bar{p})^\nu + (p+k_1-\bar{p})^\nu \frac{i}{(k_1-\bar{p})^2-m^2}(k_1-2\bar{p})^\mu$$

$$+ (-2i)\, g^{\mu\nu} \Big\}$$

d.) Substitute $k_{1\mu}$ for ε_μ (Ward-Takahashi identity)

then

$$i\mathcal{M}^{\mu\nu}k_\mu \varepsilon_{2\nu} = -e^2 \varepsilon_{2\nu} \Big[(2p\cdot k_1 - k_1^2)\frac{i}{(p-k_1)^2-m^2}(p-k_1-\bar{p})^\nu$$

$$+ (p+k_1-\bar{p})^\nu \frac{i}{(\bar{p}-k_1)^2-m^2}(k_1^2-2\bar{p}\cdot k_1) - 2i\, k_1^\nu \Big]$$

$$= -ie^2 \varepsilon_{2\nu} \Big[[p^2 - (p-k_1)^2] \frac{1}{(p-k_1)^2 - m^2} (p-k_1-\bar{p})^\nu$$

$$+ (p+k_1-\bar{p})^\nu \frac{1}{(\bar{p}-k_1)^2 - m^2} [(\bar{p}-k_1)^2 - \bar{p}^2] - 2k_1^\nu \Big]$$

since $p^2 = \bar{p}^2 = m^2$ on shell

$$= -ie^2 \varepsilon_{2\nu} \Big[(-1)(p-k_1-\bar{p})^\nu + (p+k_1-\bar{p})^\nu - 2k_1^\nu \Big]$$

$$= 0 \quad !$$

2.) $\pi^+ = \frac{1}{\sqrt{2}}(\pi^1 + i\pi^2) \qquad \pi^- = \frac{1}{\sqrt{2}}(\pi^1 - i\pi^2)$

so $\pi^1 = \frac{1}{\sqrt{2}}(\pi^+ + \pi^-) \qquad \pi^2 = \frac{-i}{\sqrt{2}}(\pi^+ - \pi^-)$

$$\pi^1\sigma^1 + \pi^2\sigma^2 + \pi^3\sigma^3$$

$$= \frac{1}{\sqrt{2}}(\pi^+ + \pi^-)\sigma^1 - \frac{i}{\sqrt{2}}(\pi^+ - \pi^-)\sigma^2 + \pi^0\sigma^3$$

$$= \frac{1}{\sqrt{2}}(\sigma^1 - i\sigma^2)\pi^+ + \frac{1}{\sqrt{2}}(\sigma^1 + i\sigma^2)\pi^- + \pi^0\sigma^3$$

$$(\sigma^1 - i\sigma^2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 2\sigma^-$$

$$(\sigma^1 + i\sigma^2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 2\sigma^+$$

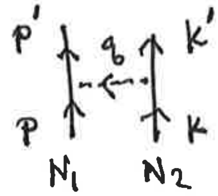
so

$$= \sqrt{2}\,\pi^+\sigma^- + \sqrt{2}\,\pi^-\sigma^+ + \pi^0\sigma^3$$

b.) The Dirac equation is $(i\gamma^\mu \partial_\mu - m) N = 0$

its conjugate is $-i\partial_\mu \bar{N} \gamma^\mu - m\bar{N} = 0$

so
$$\partial_\mu \bar{N} \sigma^i \gamma^\mu \gamma^5 N$$
$$= i(-i\partial_\mu \bar{N}\gamma^\mu)\sigma^i\gamma^5 N - i\bar{N}\sigma^i\gamma^\mu\gamma^5 i\partial_\mu N$$
$$= i(-i\partial_\mu \bar{N}\gamma^\mu)\sigma^i\gamma^5 N + i\bar{N}\sigma^i\gamma^5 i\gamma^\mu\partial_\mu N$$
$$= i \quad m_N \bar{N}\sigma^i\gamma^5 N + i\bar{N}\sigma^i\gamma^5 m_N N$$
$$= 2im_N(\bar{N}\sigma^i\gamma^5 N)$$

c.) $i\mathcal{M} =$ 

$q = p' - p = k - k'$

Expand $u(p)$ in the nonrelativistic limit
$$u(p) = \begin{pmatrix} \sqrt{p\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} = \begin{pmatrix} \sqrt{E - \vec{p}\cdot\vec{\sigma}}\,\xi \\ \sqrt{E + \vec{p}\cdot\vec{\sigma}}\,\xi \end{pmatrix}$$
$$\approx \begin{pmatrix} (\sqrt{m} - \frac{\vec{p}\vec{\sigma}}{2\sqrt{m}} + \cdots)\,\xi \\ (\sqrt{m} + \frac{\vec{p}\vec{\sigma}}{2\sqrt{m}} + \cdots)\,\xi \end{pmatrix} = \sqrt{m}\begin{pmatrix} 1 - \frac{\vec{p}\cdot\vec{\sigma}}{2m} \\ 1 + \frac{\vec{p}\vec{\sigma}}{2m} \end{pmatrix}\xi$$

similarly $\bar{u}(p) = u^\dagger(p)\gamma^0 = \sqrt{m}\,\xi^\dagger\left(1 + \frac{\vec{p}\vec{\sigma}}{2m},\ 1 - \frac{\vec{p}\vec{\sigma}}{2m}\right)$

$\gamma^5 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ so $\longrightarrow$

$$\bar{u}(q')\gamma^5 u(p) = m_N\, \xi'^{\dagger}\Big[\Big(1+\frac{\vec{p}'\cdot\vec{\sigma}}{2m_N}\Big)(-1)\Big(1-\frac{\vec{p}\cdot\vec{\sigma}}{2m_N}\Big) + \Big(1-\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\Big)(+1)\Big(1+\frac{\vec{p}\cdot\vec{\sigma}}{2m_N}\Big)\Big]\xi$$

$$= -m_N\, \xi^{\dagger}\frac{(\vec{p}'-\vec{p})\cdot\vec{\sigma}}{2m_N}\cdot 2\,\xi = -\,\xi^{\dagger}\vec{q}\cdot\vec{\sigma}\,\xi$$

For the pseudoscalar theory

$$i\mathcal{M} = g^2_{\pi NN}\,\langle N_1'|\gamma^5\sigma^i|N_1\rangle \frac{i}{q^2-m_\pi^2}\langle N_2'|\gamma^5\sigma^i|N_2\rangle$$

$$= i g^2_{\pi NN}\,(-\xi_1'^{\dagger}\vec{q}\cdot\vec{\sigma}\,\xi_1)(\sigma^i)_{N_1'N_1}\frac{-1}{|\vec{q}|^2+m_\pi^2} \cdot (+\xi_2'^{\dagger}\vec{q}\cdot\vec{\sigma}\,\xi_2)\,\sigma^i_{N_2'N_2}$$

$$= +i\, g^2_{\pi NN}\,\underbrace{\xi_1'^{\dagger}\vec{q}\cdot\vec{\sigma}\,\xi_1}_{\text{spin}}\ \underbrace{\sigma^i_{N_1'N_1}}_{\text{isospin}}\ \underbrace{\frac{1}{|\vec{q}|^2+m_\pi^2}}_{\substack{\text{Yukawa}\\ \text{potential}}}\ \underbrace{\xi_2^{\dagger}\vec{q}\cdot\vec{\sigma}\,\xi_2}_{\text{spin}}\ \underbrace{\sigma^i_{N_2'N_2}}_{\text{isospin}}$$

this interaction vanishes as $q \to 0$

d) For the pseudovector theory, we need

$$\gamma^\mu\gamma^5 = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^\mu \\ -\bar{\sigma}^\mu & 0 \end{pmatrix}$$

then

$$\bar{u}(p)\gamma^\mu\gamma^5 u(p) = m\,\xi^\dagger\Big[(1+\cdots)\,\sigma^\mu\,(1+\cdots)$$
$$+\,(1-\cdots)(-\bar{\sigma}^\mu)(1-\cdots)\Big]\xi$$

$$= m\,\xi^\dagger\cdot 2\cdot(0,\vec{\sigma}^j)^\mu\,\xi$$

That is, the $\mu=0$ term is 0 to this order, the $\mu=j$ term is

$$2m\,\xi^\dagger\vec{\sigma}^j\xi$$

For the pseudovector theory

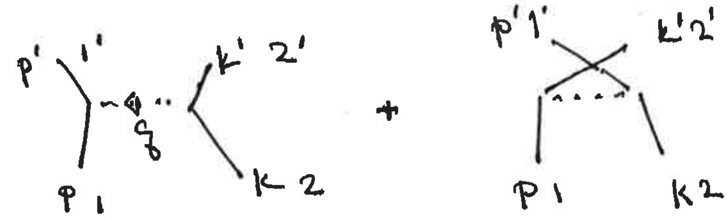
$$i\mathcal{M} = \left(\frac{g_{\pi NN}}{2m_N}\right)^2\cdot q_\mu\,\langle N_1'|\gamma^\mu\gamma^5\sigma^i|N_1\rangle\,\frac{i}{q^2-m_\pi^2}$$
$$(-q_\nu)\langle N_2'|\gamma^\nu\gamma^5|N_2\rangle$$

$$= \left(\frac{g_{\pi NN}}{2m_N}\right)^2 2m_N\,\xi_1'^\dagger(-\vec{q}\cdot\vec{\sigma})\xi_1\,(\sigma^i)_{N_1'N_1}\frac{-i}{|\vec{q}|^2+m_\pi^2}$$
$$\cdot\,2m_N\,\xi_2'^\dagger\,\vec{q}\cdot\vec{\sigma}\,\xi_2\cdot\sigma^i_{N_2'N_2}$$

$$= +i\,g_{\pi NN}^2\,\xi_1'^\dagger\,\vec{q}\cdot\vec{\sigma}\,\xi_1\,\sigma^i_{N_1'N_1}\frac{1}{|\vec{q}|^2+m_\pi^2}\,\xi_2'^\dagger\,\vec{q}\cdot\vec{\sigma}\,\xi_2\,\sigma^i_{N_2'N_2}$$

just as in part (c)

e.) $i\mathcal{M} =$



for the pseudoscalar theory:

$$i\mathcal{M} = g_{\pi NN}^2 \Big[\bar{u}(p')(\sigma^i)_{1'1} \gamma^5 u(p) \cdot \frac{i}{(p-p')^2 - m_\pi^2}$$

$$\cdot\ \bar{u}(k')(\sigma^i)_{2'2} \gamma^5 u(k)$$

$$- \ \bar{u}(k')(\sigma^i)_{2'1} \gamma^5 u(p) \ \frac{i}{(k'-p)^2 - m_\pi^2}$$

$$\bar{u}(p')(\sigma^i)_{1'2} \gamma^5 u(k) \Big]$$

f.) for the pseudovector theory, the diagrams are the same.

Their value is

$$i\mathcal{M} = \left(\frac{g_{\pi NN}}{2m_N}\right)^2 \Big[\bar{u}(p')\, \sigma^i_{1'1} (p'-p)_\mu \gamma^\mu \gamma^5 u(p) \cdot \frac{i}{(p-p')^2 - m_\pi^2}$$

$$\cdot\ \bar{u}(k')\, \sigma^i_{2'2} (k'-k)_\mu \gamma^\mu \gamma^5 u(k)$$

$$- \ \bar{u}(k')(\sigma^i)_{2'1} (k'-p)_\mu \gamma^\mu \gamma^5 u(p) \ \frac{i}{(k'-p)^2 - m_\pi^2}$$

$$\cdot\ \bar{u}(p')(\sigma^i)_{1'2} (p'-k)_\mu \gamma^\mu \gamma^5 u(k) \Big]$$

Now, $\bar{u}(p')(p'-p)_\mu \gamma^\mu \gamma^5 u(p) = \bar{u}(p') \not{p}' \gamma^5 u(p) + \bar{u}(p') \gamma^5 \not{p}\, u(p)$

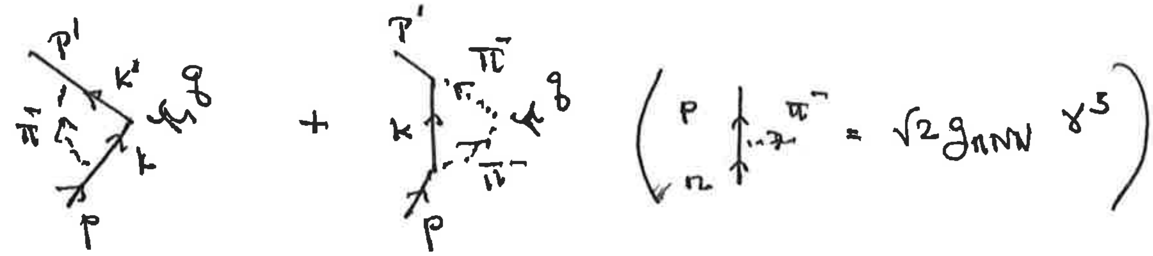
$$= 2m_N\, \bar{u}(p') \gamma^5 u(p)$$

similarly $\bar{u}(k')(k'-k)_\mu \gamma^\mu \gamma^5 u(k) = 2m_N\, \bar{u}(k') \gamma^5 u(k)$

$$\bar{u}(k')(k'-p)_\mu \gamma^\mu \gamma^5 u(p) = 2m_N\, \bar{u}(k') \gamma^5 u(p)$$

$$\bar{u}(p')(p'-k)_\mu \gamma^\mu \gamma^5 u(k) = 2m_N\, \bar{u}(p') \gamma^5 u(k)$$

with these substitutions, the expression in part (f) becomes the expression in part (e).

g.) 

$$= (\sqrt{2}\, g_{\pi NN})^2 \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} \Big[\gamma^5 \frac{i(\not{k}'+m_N)}{k'^2-m_N^2} (+ie\gamma^\mu) \frac{i(\not{k}+m_N)}{k^2-m_N^2} \gamma^5 \frac{i}{(p-k)^2-m_\pi^2}$$

$$+ \gamma^5 \frac{i(\not{k}+m)}{k^2-m^2} \gamma^5 \frac{i}{(p-k+q)^2-m_\pi^2} (-ie)(2p-2k+q)^\mu$$

$$\cdot \frac{i}{(p-k)^2-m_\pi^2} \Big] u(p)$$

Before proceeding, check the Ward Takahashi identity by dotting with q^μ:

$$q^\mu(\text{above}) = 2g_{\pi NN}^2 \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} \cdot (-i) \cdot (+ie)$$

$$\Big[\gamma^5 \left(\frac{\not{k}'+m}{(k')^2-m^2}\right) \not{q} \left(\frac{\not{k}+m}{k^2-m^2}\right) \gamma^5 \frac{1}{(p-k)^2-m_\pi^2}$$

$$- \gamma^5 \left(\frac{\not{k}+m}{k^2-m^2}\right) \gamma^5 \frac{1}{(p-k+q)^2-m_\pi^2} (2p-2k+q)\cdot q \frac{1}{(p-k)^2-m_\pi^2} \Big] u(p)$$

the term in brackets is:

$$\gamma^5 \frac{(\not{k}+\not{q})+m}{(k+q)^2-m^2} \big(\underbrace{\not{k}+\not{q}}_{-m} - \underbrace{\not{k}}_{+m}\big) \left(\frac{\not{k}+m}{k^2-m^2}\right) \gamma^5 \frac{1}{(p-k)^2-m_\pi^2}$$

$$- \gamma^5 \left(\frac{\not{k}+m}{k^2-m^2}\right) \gamma^5 \frac{1}{(p-k+q)-m_\pi^2} \left[(p-k+q)^2-(p-k)^2\right] \frac{1}{(p-k)^2-m_\pi^2}$$

$$= \left(\gamma^5 \frac{\not k + m}{k^2 - m^2} \gamma^5 - \gamma^5 \frac{\not k + \not q + m}{(k+q)^2 - m^2} \gamma^5 \right) \frac{1}{(p-k)^2 - m_\pi^2}$$

$$- \gamma^5 \left(\frac{\not k + m}{k^2 - m^2}\right) \gamma^5 \left(\frac{1}{(p-k)^2 - m_\pi^2} - \frac{1}{(p-k+q)^2 - m_\pi^2} \right)$$

$$= \gamma^5 \left[- \frac{(\not k + \not q + m)}{(k+q)^2 - m^2} \frac{1}{(p-k)^2 - m_\pi^2} + \underbrace{\frac{\not k + m}{k^2 - m^2} \frac{1}{(p-k+q)^2 - m_\pi^2}} \right] \gamma^5$$

vanishes after the shift $k \to (k+q)$ in this term

$$= 0$$

so we can have confidence in the expression on p. 9, Slightly simplifying

$$- e (\sqrt{2})^2 g_{\pi NN}^2 \int \frac{d^4 k}{(2\pi)^4} \bar{u}(p')$$

(I) $$\left[\frac{\gamma^5 (\not k + \not q + m) \gamma^\mu (\not k + m) \gamma^5}{((k+q)^2 - m^2)(k^2 - m^2)((p-k)^2 - m_\pi^2)} \right.$$

(II) $$\left. - \frac{\gamma^5 (\not k + m) \gamma^5 (2p - 2k + q)^\mu}{(k^2 - m^2)((p-k)^2 - m_\pi^2)((p-k+q)^2 - m_\pi^2)} \right] u(p)$$

h.) Now simplify (I) and (II). For (I) introduce Feynman parameters.

$$x \left[(k+q)^2 - m_N^2 \right] + y \left[k^2 - m^2 \right] + z \left[(p-k)^2 - m_\pi^2 \right]$$

$$= k^2 + 2x\, k\cdot q + x q^2 - 2z\, k\cdot p + z p^2 - (1-z) m_N^2 - z m_\pi^2$$

$$\ell = k + xq - zp$$

then $k = \mathbb{k} - xq + zp$

$k+q = \mathbb{k} + (1-x)q + zp$

the denominator becomes

$$\text{Den} = \mathbb{k}^2 + x\overbrace{(1-x)}^{y+z} q^2 + z\overbrace{(1-z)}^{x+y} p^2 + 2xz\, q\cdot p - (1-z)m_N^2 - z m_\pi^2$$

$$= \mathbb{k}^2 + xz(p+q)^2 + yz\, p^2 + xy\, q^2 - (1-z)m_N^2 - z m_\pi^2$$

$$= \mathbb{k}^2 + xy\, q^2 + z(1-z)m_N^2 - (1-z)m_N^2 - z m_\pi^2$$

$$= \mathbb{k}^2 + xy\, q^2 - (1-z)^2 m_N^2 - z m_\pi^2$$

$$= \mathbb{k}^2 - \Delta \qquad \Delta = (1-z)^2 m_N^2 + z m_\pi^2 - xy\, q^2$$

the numerator is

$$\bar{u}(p')\left[\gamma^5(\not{\mathbb{k}} + (1-x)\not{q} + z\not{p} + m)\gamma^\mu(\not{\mathbb{k}} - x\not{q} + z\not{p} + m_N)\gamma^5 u(p)\right.$$

γ^5 anticommutes w. γ^μ

$$= \bar{u}(p')\left[(-1)(\not{\mathbb{k}} + \overbrace{(1-x)}^{y+z}\not{q} + z\not{p} - m_N)\gamma^\mu(\not{\mathbb{k}} - x\not{q} + z\not{p} - m_N)\right] u(p)$$

$$= -\bar{u}(p')\left[(\not{\mathbb{k}} + y\not{q} + z\not{p}' - m_N)\gamma^\mu(\not{\mathbb{k}} - x\not{q} + z\not{p} - m_N)\right] u(p)$$

$$= -\bar{u}(p')\left[(\not{\mathbb{k}} + y\not{q} - (1-z)m_N)\gamma^\mu(\not{\mathbb{k}} - x\not{q} - (1-z)m_N)\right] u(p)$$

$$= -\bar{u}(p')\left[\not{\mathbb{k}}\gamma^\mu\not{\mathbb{k}} + (\text{linear in } \mathbb{k}) - xy\,\not{q}\gamma^\mu\not{q} - x(1-z)m_N\gamma^\mu\not{q} + y(1-z)m_N\not{q}\gamma^\mu + (1-z)^2 m_N^2\gamma^\mu\right] u(p)$$

$$= -\bar{u}(p')\Big[\,\not{D}\!\!\!\!\not{k}\,\gamma^\mu\,\not{D}\!\!\!\!\not{k} - xy\left(\not{q}\cdot 2q^\mu - q^2\gamma^\mu\right)$$
$$-x(1-z)\,m_N\,\left[q^\mu + \tfrac{1}{2}[\gamma^\mu,\not{q}]\right]$$
$$+y(1-z)\,m_N\,\left[q^\mu - \tfrac{1}{2}[\gamma^\mu,\not{q}]\right] + (1-z)^2 m_N^2\gamma^\mu\Big]u(p)$$

$$\bar{u}(p')\,\not{q}\,u(p) = \bar{u}(p')(\not{p}'-\not{p})u(p) = 0$$

$\langle x\rangle = \langle y\rangle = \frac{1}{2}\langle(1-z)\rangle$ so the q^μ terms cancel.

$$= -\bar{u}(p')\left[\not{D}\!\!\!\!\not{k}\,\gamma^\mu\,\not{D}\!\!\!\!\not{k} + xy\,\gamma^\mu q^2 + i\sigma^{\mu\nu}q_\nu(1-z)^2 m_N + (1-z)^2 m_N^2\gamma^\mu\right]u(p)$$

in all,

$$(\mathrm{I}) = +2e\,g_{\pi NN}^2\int dx\,dy\,dz\,\delta(x+y+z-1)\cdot 2$$
$$\cdot\int\frac{d^dk}{(2\pi)^d}\,\frac{\bar{u}(p')\left[\not{D}\!\!\!\!\not{k}\,\gamma^\mu\,\not{D}\!\!\!\!\not{k} + xy\,\gamma^\mu q^2 + (1-z)^2 m_N^2\gamma^\mu + i\sigma^{\mu\nu}q_\nu(1-z)^2 m_N\right]u}{[\mathcal{D}k^2-\Delta]^3}$$

the contribution to F_2 is

$$F_2(\mathrm{I}) = 2e\,g_{\pi NN}^2\int dx\,dy\,dz\,\delta(x+y+z-1)\cdot 2$$
$$\cdot\int\frac{d^dk}{(2\pi)^d}\,\frac{(1-z)^2\,2m_N^2}{[\mathcal{D}k^2-\Delta]^3}$$

and the rest contributes to F_1

For (II) introduce Feynman parameters

$$z\left(k^2-m_N^2\right) + x\left((k-p)^2-m_\pi^2\right) + y\left((k-p-q)^2-m_\pi^2\right)$$

$$= k^2 - 2x\, k\cdot p + xp^2 - 2y\, k\cdot(p+q) + y(p-q)^2$$
$$- z m_N^2 - (1-z)\, m_\pi^2$$

so $\mathbb{k} = k - xp - y(p+q)$

$$k = \mathbb{k} + xp + yp'$$

$$2p - 2k + q = -2k + p + p' = -2\mathbb{k} + (1-2x)p + (1-2y)p'$$

the denominator is

$$\text{Den} = \mathbb{k}^2 + x\underbrace{(1-x)}_{y+z}p^2 + y\underbrace{(1-y)}_{x+z}(p')^2 - 2xy\, p\cdot p'$$
$$- z m_N^2 - (1-z)m_\pi^2$$

$$= \mathbb{k}^2 + xy(p-p')^2 + xz\, m_N^2 + yz\, m_N^2 - z m_N^2 - (1-z)m_\pi^2$$

$$= \mathbb{k}^2 + xy\, q^2 - z(1-x-y)m_N^2 - (1-z)\, m_\pi^2$$

$$= \mathbb{k}^2 - \Delta' \qquad \Delta' = z^2 m_N^2 + (1-z)\, m_\pi^2 - xy\, q^2$$

the numerator is

$$-\bar{u}(p')\,\gamma^5\left(\not{\mathbb{k}} + x\not{p} + y\not{p}' + m\right)\gamma^5\, u(p)\cdot\left(-2\mathbb{k} + (1-2x)p + (1-2y)p'\right)^\mu$$

$$= +\,\bar{u}(p')\left[\not{\mathbb{k}} + x\not{p} + y\not{p}' - m_N\right]u(p)\left(-2\mathbb{k} + (1-2x)p + (1-2y)p'\right)^\mu$$

$$= +\,\bar{u}(p')\left[\not{\mathbb{k}} - z m_N\right]u(p)\left(-2\mathbb{k}^\mu + (1-2x)p^\mu + (1-2y)p'^\mu\right)$$

under the ~~integral~~ over Feynman parameters

$$\langle 1-2x \rangle = \langle 1-2y \rangle = \langle 1-(x+y) \rangle = \langle z \rangle$$

$$= \bar{u}(p') \left[\not{k} - z m_N \right] \left(-2k^\mu + z\, p^\mu + p'^\mu \right) u(p)$$

$$= \bar{u}(p') \left[-2\not{k}\, k^\mu - z^2 m_N (p+p')^\mu \right] u(p) + (\text{linear in } k)$$

and use the Gordon identity

$$= \bar{u}(p') \left[-2\not{k}\, k^\mu - 2z^2 m_N^2 \gamma^\mu + i z^2 2m_N^2 \frac{\sigma^{\mu\nu} q_\nu}{2m_N} \right] u(p)$$

in all

$$\text{II} = +2e\, g_{\pi NN}^2 \int dx\, dy\, dz\; \delta(x+y+z-1) \cdot 2$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{\bar{u}(p') \left[+2\not{k}\, k^\mu + 2z^2 m_N^2 \gamma^\mu - 2m_N^2 z^2 \frac{i\sigma^{\mu\nu} q_\nu}{2m_N} \right] u(p)}{\left[k^2 - \Delta' \right]^3}$$

the contribution to F_2 is

$$F_2(\text{II}) = 2e\, g_{\pi NN}^2 \int dx\, dy\, dz\; \delta(x+y+z-1) \cdot 2$$

$$\int \frac{d^d k}{(2\pi)^d} \left(\frac{-z^2\, 2m_N^2}{\left[k^2 - \Delta' \right]^3} \right)$$

combine the two contributions to F_2 using

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\left[k^2 - \Delta \right]^3} = \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(3-d/2)}{\underbrace{\Gamma(3)}_{=2} \Delta^{3-d/2}}$$

$$d=4 \qquad = \frac{-i}{(4\pi)^2} \frac{1}{2} \frac{1}{\Delta} \qquad (\text{finite})$$

$$(+ie)\, F_2(q^2) = (-i)\frac{2e\, g_{\pi NN}^2}{(4\pi)^2}\int dx\, dy\, dz\, \delta(x+y+z-1)$$

$$\left(\frac{2m_N^2(1-z)^2}{[(1-z)^2 m_N^2 + z m_\pi^2 - xyq^2]} - \frac{2m_N^2 z^2}{[z^2 m_N^2 + (1-z)m_\pi^2 - xyq^2]}\right)$$

$$F_2(q^2) = -\frac{g_{\pi NN}^2}{8\pi^2}\int dx\, dy\, dz\, \delta(x+y+z-1)$$

$$\left(\frac{2m_N^2(1-z)^2}{(1-z)^2 m_N^2 + z m_\pi^2 - xyq^2} - \frac{2m_N^2 z^2}{z^2 m_N^2 + (1-z)m_\pi^2 - xyq^2}\right)$$

the denominator is positive for $q^2 \le 0$ so this is also IR finite. For $q^2 = 0$

$$\int dx\, dy\, dz\, \delta(\,) = \int_0^1 dz \int_0^{1-z} dx = \int_0^1 dz\,(1-z)$$

$$F_2(0) = -\frac{g_{\pi NN}^2}{8\pi^2}\int_0^1 dz\,(1-z)\, 2m_N^2\left(\frac{(1-z)^2}{[(1-z)^2 m_N^2 + z m_\pi^2)} - \frac{z^2}{(1-z)^2 m_N^2 + z m_\pi^2}\right)$$

$$\underline{\qquad} \quad z \to (1-z)$$

$$= \frac{g_{\pi NN}^2}{8\pi^2}\int_0^1 dz\left(\frac{2m_N^2(1-2z)}{z^2 m_N^2 + (1-z)m_\pi^2}\right)$$

i) Now work on $F_1(0)$, still using dimensional regularization

we need also

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{[k^2-\Delta]^3} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\underbrace{\Gamma(3)}_{=2}\, \Delta^{2-d/2}} \cdot \frac{g^{\mu\nu}}{2}$$

$$(+ie)F_1(0)\gamma^\mu = \frac{2ie\, g_{\pi NN}^2}{(4\pi)^{d/2}} \int dx\,dy\,dz\; \delta(x+y+z-1)$$

$$\left(\Gamma(2-d/2)\left[\frac{1}{2}\frac{(2-d)\gamma^\mu}{\Delta^{2-d/2}} + \frac{3}{2}\frac{\gamma^\mu}{[\Delta']^{2-d/2}} \right] \right.$$

$$\left. - \Gamma(3-d/2)\left[\frac{(1-z)^2 m_N^2 \gamma^\mu}{\Delta^{3-d/2}} + \frac{2z^2 m_N^2 \gamma^\mu}{[\Delta']^{3-d/2}} \right] \right)$$

$$F_1(0) = \frac{g_{\pi NN}^2}{8\pi^2}\, \Gamma[3-d/2] \int dx\,dy\,dz\; \delta(x+y+z-1)$$

$$\left\{ \left(\frac{1-d/2}{2-d/2}\frac{1}{\Delta^{2-d/2}} + \frac{1}{2-d/2}\frac{1}{[\Delta']^{2-d/2}} \right) \right.$$

$1-d/2 = (2-d/2)-1$

$$\left. - \left(\frac{(1-z)^2 m_N^2}{\Delta^{3-d/2}} + \frac{2z^2 m_N^2}{[\Delta']^{3-d/2}} \right) \right\}$$

Notice that the pole at $d=4$ cancels in the 1^{st} line; this quantity is UV (and IR) finite!

the integrand depends only on z

$$F_1(0) = \frac{g^2_{\pi NN}}{8\pi^2}\,\Gamma(3-d/2)\int_0^1 dz\,(1-z)$$

$$\left\{\left(1-\frac{1}{2-d/2}\right)\frac{1}{[(1-z)^2m_N^2+zm_\pi^2]^{2-d/2}}+\frac{1}{(2-d/2)}\frac{1}{[z^2m_N^2+(1-z)m_\pi^2]^{2-d/2}}\right.$$

$$\left.-\frac{(1-z)^2m_N^2}{[(1-z)^2m_N^2+zm_\pi^2]^{3-d/2}}-\frac{2z^2m_N^2}{[z^2m_N^2+(1-z)m_\pi^2]^{3-d/2}}\right\}$$

$\uparrow$ Let $w=(1-z)$ in these terms $\qquad$ $\uparrow$ Let $w=z$ in these terms.

$$=\frac{g^2_{\pi NN}}{8\pi^2}\,\Gamma(3-d/2)\int_0^1 dw$$

$$\left\{\left(1-\frac{1}{2-d/2}\right)\frac{w}{[w^2m_N^2+(1-w)m_\pi^2]^{2-d/2}}+\frac{1}{2-d/2}\frac{(1-w)}{[w^2m_N^2+(1-w)m_\pi^2]^{2-d/2}}\right.$$

$$\left.-\frac{w^3m_N^2}{[w^2m_N^2+(1-w)m_\pi^2]^{3-d/2}}-\frac{2w^2(1-w)m_N^2}{[w^2m_N^2+(1-w)m_\pi^2]^{3-d/2}}\right\}$$

$$=\frac{g^2_{\pi NN}}{8\pi^2}\,\Gamma(3-d/2)\int_0^1 dw$$

$$\left\{\frac{1}{(2-d/2)}\frac{1-2w}{[w^2m_N^2+(1-w)m_\pi^2]^{2-d/2}}+\frac{w}{[w^2m_N^2+(1-w)m_\pi^2]^{2-d/2}}\right.$$

$$\left.-\frac{w^3m_N^2+2w^2(1-w)m_N^2}{[w^2m_N^2+(1-w)m_\pi^2]^{3-d/2}}\right\}$$

Integrate by parts in the first term

$$= \frac{g_{\pi NN}^2}{8\pi^2}\,\Gamma(3-d/2)\left\{ \frac{(w-w^2)}{2-d/2}\,\frac{1}{\left[w^2 m_N^2+(1-w)\,m_\pi^2\right]^{2-d/2}}\right|_0^1$$

$$+\int_0^1 dw\left\{ \frac{(w-w^2)(2w m_N^2-m_\pi^2)}{\left[w^2 m_N^2+(1-w)\,m_\pi^2\right]^{3-d/2}} + \frac{w(w^2 m_N^2+(1-w)\,m_\pi^2)}{\left[w^2 m_N^2+(1-w)\,m_\pi^2\right]^{3-d/2}} - \frac{w^3 m_N^2+2w^2(1-w)\,m_N^2}{\left[w^2 m_N^2+(1-w)\,m_\pi^2\right]^{3-d/2}}\right\}$$

$$= \frac{g_{\pi NN}^2}{8\pi^2}\,\Gamma(3-d/2)\Big\{\ 0$$

$$+\int_0^1 dw\,\frac{1}{\left[w^2 m_N^2+(1-w)\,m_\pi^2\right]^{3-d/2}}$$

$$\left\{ 2w^2 m_N^2-2w^3 m_N^2-\cancel{w(1-w)}\,m_\pi^2+w^3 m_N^2+\cancel{w(1-w)}\,m_\pi^2 - w^3 m_N^2 - 2w^2 m_N^2+2w^3 m_N^2 \right\}$$

$$\{\} = m_N^2\left\{\cancel{2w^2}-2w^3+w^3-w^3-\cancel{2w^2}+2w^3\right\}$$

$$= 0\ !$$

so $F_1(0) = 0$ in any dimension d, as required

j) For the diagrams on p. 9, their value in the pseudovector model is

$$\left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \bar{u}(p') \int \frac{d^4k}{(2\pi)^4}$$

$$\Big[(\not p' - \not k')\gamma^5 \frac{i(\not k' + m_N)}{(k')^2 - m^2} (+ie\gamma^\mu) \frac{i(\not k + m_N)}{k^2 - m_N^2} (\not k - \not p)\gamma^5 \cdot \frac{i}{(p-k)^2 - m_\pi^2}$$

$$+ (\not p' - \not k)\gamma^5 \frac{i\,\not k + m_N}{k^2 - m_N^2} (\not k - \not p)\gamma^5 \frac{i}{(p-k+q)^2 - m_\pi^2} (-ie)(2p - 2k + q)^\mu \frac{i}{(p-k)^2 - m_\pi^2}\Big]$$

$$\cdot\, u(p)$$

In the 1st term

$$\bar{u}(p')(\not p' - \not k')\gamma^5 (k' + m_N)$$

$$= \bar{u}(p')\left(\not p'\gamma^5 + \gamma^5(\not k' - m_N + m_N)\right)(k' + m_N)$$

$$= 2m_N \underbrace{\bar{u}(p')\gamma^5(k' + m)}_{\text{contributes in the pseudoscalar case}} + \bar{u}(p')\gamma^5 \underbrace{(k'^2 - m_N^2)}_{\text{cancels the first denominator}}$$

similarly

$$(\not k + m)(\not k - \not p)\gamma^5 u(p) = 2m_N(\not k + m_N)\gamma^5 u(p) + (k^2 - m_N^2)\gamma^5 u(p)$$

and in the 2nd diagram

$$\bar{u}(p')(p' - k)\gamma^5(\not k + m_N) = 2m_N\, \bar{u}(p')\gamma^5(\not k + m_N) + \bar{u}(p')\gamma^5(k^2 - m_N^2)$$

$$(\not k + m_N)(\not k - \not p)\gamma^5 u(p) = 2m_N(\not k + m_N)u(p) + (k^2 - m_N^2)\gamma^5 u(p)$$

in all cases, the first term $\times \left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)$ gives

$$\sqrt{2}\, g_{\pi NN} \cdot (\text{pseudoscalar result})$$

The remaining terms are the difference between the pseudovector and pseudoscalar result.

so

(pseudovector) − (pseudoscalar)

$$= (\pm ie) \left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 (i)^3 \int \frac{d^4k}{(2\pi)^4} \quad \bar{u}(p')$$

$$\Big\{ \Big[\gamma^5 \gamma^\mu \left(\frac{\not{k}+m_N}{k^2-m_N^2}\right) 2m_N \gamma^5 + \gamma^5\, 2m_N \left(\frac{\not{k}'+m_N}{k'^2-m_N}\right) \gamma^\mu \gamma^5$$

$$+ \gamma^5 \gamma^\mu \gamma^5 \Big] \frac{1}{(p-k)^2-m_\pi^2}$$

$$- \Big[\gamma^5\, 2m_N \gamma^5 + \gamma^5\, 2m_N \gamma^5 + \gamma^5 (\not{k}-m_N) \gamma^5 \Big]$$

$$\cdot \frac{1}{(p-k+q)^2-m_\pi^2} (2p-2k+q)^\mu \frac{1}{(p-k)^2-m_\pi^2} \Big\} u(p)$$

If you got this far, you have answered everything that I asked on the exam. However, it is interesting to go a little further.

the results on p. 20 have only 2 denominators, so they are intrinsically easier to deal with that the full vertex corrections. Let's evaluate them using dimensional regularization:

First, do the integrals that we need:

$$\int \frac{d^dk}{(2\pi)^d} \frac{\not{k}+m_N}{k^2-m_N^2} \frac{1}{(k-p)^2-m_\pi^2}$$

$$= \int_0^1 dx \int \frac{d^dk}{(2\pi)^4} \frac{\not{k}+m_N}{\left[k^2-2x\,k\cdot p+xp^2-(1-x)m_N^2-xm_\pi^2\right]^2}$$

$$\mathbb{k} = k - xp \qquad k = \mathbb{k} + xp$$

$$= \int_0^1 dx \int \frac{d^d\mathbb{k}}{(2\pi)^d} \frac{\not{\mathbb{k}} + x\not{p} + m_N}{\left[\mathbb{k}^2 + x(1-x)p^2 - (1-x)m_N^2 - xm_\pi^2\right]^2}$$

$$= \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)\ \Delta^{2-d/2}} (x\not{p}+m_N)$$

where $$\Delta = (1-x)m_N^2 + xm_\pi^2 - x(1-x)m_N^2$$

$$= (1-x)^2 m_N^2 + xm_\pi^2$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{\not{k}'+m}{[(k')^2-m_N^2][(k-p)^2-m_\pi^2]} \qquad k' = k+q$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{(\not{k}+\not{q}+m)}{[(k+q)^2-m_N^2][(k+q-(p+q))^2-m_\pi^2]}$$

$$= \int \frac{d^dk'}{(2\pi)^d} \frac{(\not{k}'+m)}{[(k')^2-m_N^2][(k'-p')^2-m_\pi^2]}$$

$$= \int_0^1 dx \, \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)\,\Delta^{2-d/2}} (x\not{p}' + m_N)$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k-p)^2 - m_\pi^2} = \int \frac{d^d\bar{k}}{(2\pi)^d} \frac{1}{\bar{k}^2 - m_\pi^2} = \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)}{\Gamma(1)\,[m_\pi^2]^{1-d/2}}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(p-k+q)^2 - m_\pi^2} (2p-2k+q)^\mu \frac{1}{(p-k)^2 - m_\pi^2} \qquad \bar{k} = (p-k), \quad k = p - \bar{k}$$

$$= \int \frac{d^d\bar{k}}{(2\pi)^d} \frac{1}{(\bar{k}+q)^2 - m_\pi^2} (2\bar{k}+q)^\mu \frac{1}{\bar{k}^2 - m_\pi^2}$$

$$= \int_0^1 dx \int \frac{d^d\bar{k}}{(2\pi)^d} \frac{(2\bar{k}+q)^\mu}{[\bar{k}^2 + 2x\bar{k}\cdot q + xq^2 - m_\pi^2]^2} \qquad \mathbb{k} = \bar{k} + xq, \quad \bar{k} = \mathbb{k} - xq$$

$$= \int_0^1 dx \int \frac{d^d\mathbb{k}}{(2\pi)^d} \frac{(2\mathbb{k}^\mu + (1-2x)q^\mu)}{[\mathbb{k}^2 + x(1-x)q^2 - m_\pi^2]^2}$$

$$= \int_0^1 dx \, \frac{i}{(4\pi)^{d/2}} (1-2x) \frac{\Gamma(2-d/2)}{\Gamma(2)\,[m_\pi^2 - x(1-x)q^2]^{2-d/2}}$$

but $1-2x = (1-x) - x$ antisymmetric in $x \leftrightarrow 1-x$

so this $= 0$

$$\int \frac{d^dk}{(2\pi)^d} \frac{(2p-2k+q)^\mu (\not{k} - m_N)}{((\bar{k}+q)^2 - m_\pi^2)(\bar{k}^2 - m_\pi^2)}$$

$$= \int \frac{d^d\bar{k}}{(2\pi)^d} \frac{(2\bar{k}+q)^\mu (\not{p} - \not{\bar{k}} - m_N)}{((\bar{k}+q)^2 - m_\pi^2)(\bar{k}^2 - m_\pi^2)}$$

$$= \int_0^1 dx \int \frac{d^d\bar{k}}{(2\pi)^d} \frac{(2\mathbb{k}^\mu + (1-2x)q^\mu)(\not{p} - \not{\mathbb{k}} + x\not{q} - m_N)}{[\mathbb{k}^2 - \bar{\Delta}]^2} \qquad \bar{\Delta} = m_\pi^2 - x(1-x)q^2$$

$$= \int_0^1 dx \left\{ \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)}{\Gamma(1)[\bar{\Delta}]^{1-d/2}} \gamma^\mu + \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)\,\bar{\Delta}^{2-d/2}} (1-2x) q^\mu (\not{p} - x\not{q} - m_N) \right\}$$

use these results and anti commuting nature of γ^5 to evaluate p. 20

(PV) − (PS)

$$= (+ie)\left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \bar{u}(p') \int_0^1 dx \frac{1}{(4\pi)^{d/2}}$$

$$\left\{ \frac{\Gamma'(2-d/2)}{\Delta^{2-d/2}} \left[2m_N \gamma^\mu (x\not{p} - m_N) + 2m_N (x\not{p}' - m_N)\gamma^\mu \right] \right.$$

$$+ \frac{\Gamma(1-d/2)}{[m_\pi]^{1-d/2}} \gamma^\mu + 0$$

$$\left. + \frac{\Gamma(1-d/2)}{[\bar{\Delta}]^{1-d/2}} \gamma^\mu - \frac{\Gamma(2-d/2)}{[\bar{\Delta}]^{2-d/2}} (\not{p} - x\not{q} + m_N)(1-2x) q^\mu \right\}$$

$$\cdot\, u(p)$$

$$= (+ie)\left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \frac{1}{(4\pi)^{d/2}} \bar{u}(p')$$

$$\left\{ \left(\frac{\Gamma(1-d/2)}{[m_\pi^2]^{1-d/2}} + \frac{\Gamma(1-d/2)}{[m_\pi^2 - x(1-x)q^2]^{1-d/2}} \right) \gamma^\mu \right.$$

$$+ \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left(-2m_N^2(1-x)\gamma^\mu\right) \cdot 2$$

$$\left. - \frac{\Gamma(2-d/2)}{[\bar{\Delta}]^{2-d/2}} (2m_N) \underbrace{(1-2x)}_{\text{integrates to } 0} q^\mu \right\} u(p)$$

this only contributes to $F_1(q^2)$

$$= (+ie)\left(\frac{\sqrt{2}\, g_{\omega NN}}{2m_N}\right)^2 \frac{1}{(4\pi)^{d/2}}\ \bar{u}(p')\gamma^\mu u(p)$$

$$\cdot \frac{1}{(4\pi)^{d/2}} \int_0^1 dx \left\{ \Gamma(1-d/2)\left(\frac{1}{(m_\pi^2)^{1-d/2}} + \frac{1}{[m_\pi^2 - x(1-x)q^2]^{1-d/2}}\right)\right.$$

$$\left. - \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\left(4m_N^2(1-x)\right)\right\} \qquad \Delta = (1-x)^2 m_N + x\, m_\pi^2$$

but, ① this is not $= 0$ at $q^2 = 0$

② there is a quadratic divergence $\Gamma(1-d/2)$ that does not cancel.

The reason for this is that there are some additional diagrams that must be included. These terms were known in 1949, but I did not explain them to you in Physics 330.

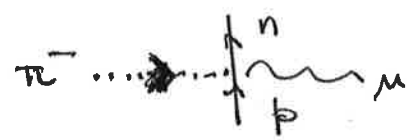
In order for eq. (12) in the problem set to respect gauge invariance or current conservation, some extra terms should be added. Specifically, when $\partial_\mu \pi$ appears in the ΔH, we must replace ∂_μ by $(\partial_\mu + ie A_\mu Q) = D_\mu$ where Q is the electric charge. This is the principle of "minimal coupling" that you might have seen in your quantum mechanics course.

so eq.(12) should really have been:

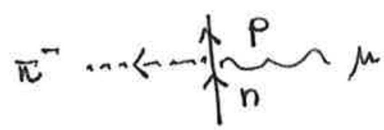
$$\Delta H = \int d^3x \left(-\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)\Big\{\big[(\partial_\mu + ieA_\mu)\pi^+\big]\, \bar{n}\gamma^\mu\gamma^5 p + \big[(\partial_\mu - ieA_\mu)\pi^-\big]\, \bar{p}\gamma^\mu\gamma^5 n\Big\}$$

(for the electron field, we have $\partial_\mu \psi_e \to (\partial_\mu + ieA_\mu)\psi_e$ where ψ_e is the field that destroys an electron. In this case, $\pi^+(x)$ is the field that destroys a π^-. This explains which field gets a $+ie$ and which gets a $-ie$.)

The new terms give rise to 2 more vertices

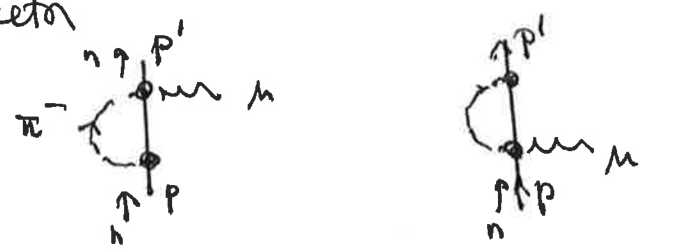


$$= \left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)(-e)\,\gamma^\mu\gamma^5$$



$$= \frac{\sqrt{2}\, g_{\pi NN}}{2m_N}(+e)\,\gamma^\mu\gamma^5$$

and thus there are 2 more diagrams contributing to the neutron form factor



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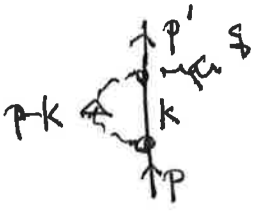


$$= \left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \int \frac{d^4k}{(2\pi)^4}$$

$$\bar{u}(p)\Big[-e\gamma^\mu\gamma^5 \; i\left(\frac{\not{k}+m_N}{k^2-m_N^2}\right) \underbrace{(\not{k}-\not{p})}_{\{(\not{k}-m_N+m_N)\gamma^5 + m_N\gamma^5\}}\gamma^5\Big]u(p)\; \frac{i}{(p-k)^2-m_\pi^2}$$

$$= (+e)\left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \int \frac{d^4k}{(2\pi)^4}$$

$$\cdot\, \bar{u}(p')\left[\gamma^\mu \frac{1}{(p-k)^2-m_\pi^2} + \gamma^\mu \frac{(-\not{k}+m_N)}{k^2-m_N^2}\cdot 2m_N \frac{1}{((p-k)^2-m_\pi^2)}\right] u(p)$$

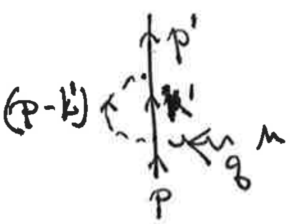
$$= e\left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2$$ (using the previously computed integrals)

$$\cdot \Big\{ \bar{u}(p')\gamma^\mu u(p) \cdot \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)}{(m_\pi^2)^{1-d/2}}$$

$$+ \bar{u}(p') \int_0^1 dx\, \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \gamma^\mu(-x\not{p} + m_N)\, 2m_N \Big\}\, u(p)$$

$$= (+ie)\left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \left(\bar{u}\gamma^\mu u(p)\right) \frac{1}{(4\pi)^{d/2}}$$

$$\cdot \int_0^1 dx \left\{ -\frac{\Gamma(1-d/2)}{[m_\pi]^{1-d/2}} + \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\, 2m_N^2 (1-x) \right\}$$



$$= \left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \int \frac{d^4k}{(2\pi)^4}$$

$$\bar{u}(p')(\not{p}'-\not{k}')\gamma^5 \frac{i(\not{k}'+m_N)}{((k')^2-m_N^2)} (+e)\, \gamma^\mu\gamma^5\, u(p) \cdot \frac{i}{(p-k')^2-m_\pi^2}$$

$$= (-e)\left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \int \frac{d^4k}{(2\pi)^4}$$

$$\bar{u}(p')\left[2m_N\gamma^5 + \gamma^5(\not{k}'-m_N)\right]\left(\frac{\not{k}'+m_N}{(k')^2-m_N^2}\right)\gamma^\mu\gamma^5\, u(p)\, \frac{1}{(p-k')^2-m_\pi^2}$$

$$= (-e)\left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \int \frac{d^4k}{(2\pi)^4}$$

$$\bar{u}(p')\left[\, 2m_N \gamma^5 \frac{\not{k}' + m_N}{k'^2 - m_N^2} \gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu \gamma^5 \right] u(p) \frac{1}{(k'-p)^2 - m_\pi^2}$$

$$= (-e)\left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \int \frac{d^4k'}{(2\pi)^4}$$

$$\left\{ -\bar{u}(p')\gamma^\mu u(p) \frac{1}{(k')^2 - m_\pi^2} + \bar{u}(p')\, 2m_N \frac{\not{k}' - m_N}{k'^2 - m_N^2\ (k'-p)^2 - m_\pi^2} \gamma^\mu u(p) \right\}$$

$$= +e \left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2$$

$$\int_0^1 dx \left\{ u(p')\gamma^\mu u(p) \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)}{(m_\pi^2)^{1-d/2}} - \frac{i}{(4\pi)^{d/2}} \bar{u}(p') \frac{\Gamma(2-d/2)\,(x\not{p}' - m_N)}{\Delta^{2-d/2}} \cdot 2m_N \gamma^\mu \right\} u(p)$$

$$= (+ie)\left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \bar{u}\gamma^\mu u \frac{1}{(4\pi)^{d/2}}$$

$$\int_0^1 dx \left\{ -\frac{\Gamma(1-d/2)}{(m_\pi^2)^{1-d/2}} + \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\, 2m_N^2 (1-x) \right\}$$

in all $\rightarrow$

$F_1(PV) - F_1(PS)$

$$= \left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \frac{1}{(4\pi)^{d/2}}$$

$$\cdot \int_0^1 dx \Big\{ \Gamma(1-d/2) \left[\frac{1}{[m_\pi^2]^{1-d/2}} + \frac{1}{[m_\pi^2 - x(1-x)q^2]^{1-d/2}} \right]$$

$$\cancel{- \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}} \; 4m_N^2(1-x)$$ p.24

$$- 2\frac{\Gamma(1-d/2)}{[m_\pi^2]^{1-d/2}} + \cancel{\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}} \; 4m_N^2(1-x) \Big\}$$ p.26,27

$$= \left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \frac{1}{(4\pi)^{d/2}} \cdot \int_0^1 dx \; \Gamma(1-d/2) \left[\frac{1}{[m_\pi^2 - x(1-x)q^2]^{1-d/2}} - \frac{1}{(m_\pi^2)^{1-d/2}} \right]$$

note that the poles at $d=2$ cancel! So, there is no quadratic divergence. As $d \to 4$

$$= \left(\frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\right)^2 \frac{1}{(4\pi)^{d/2}} \int_0^1 dx \; \frac{\Gamma(2-d/2)}{(1-d/2)} \left(\frac{[m_\pi^2 - x(1-x)q^2)}{[m_\pi^2 - x(1-x)q^2]^{2-d/2}} - \frac{m_\pi^2}{[m_\pi^2]^{2-d/2}} \right)$$

$$= \frac{2g_{\pi NN}^2}{4m_N^2} \frac{1}{(4\pi)^{d/2}} \int_0^1 dx \; \frac{\Gamma(2-d/2)}{(1-d/2)}$$

$$\cdot \left[-x(1-x)q^2 + \mathcal{O}(2-d/2) \right)$$

so $F_1^{(PV)}(q^2=0) = 0$ as required

$F_1(PV) - F_1(PS)$

$$= \frac{g^2_{\pi NN}}{32\pi^2 m_N^2} \cdot \left(-\frac{1}{6}\right) q^2 \cdot \left(-I(2\ d/2\right) + \text{finite}$$

or

$$F_1(PV) - F_1(PS) = \frac{g^2_{\pi NN}}{192\pi^2} \frac{q^2}{m_N^2} \lg \Lambda^2/m_\pi^2 + \text{finite}$$

So there is a difference in the order q^2 term which is log-divergent

The integrals for the finite piece are not very difficult; Feynman could well have done this analysis in 1 evening.