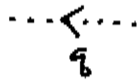
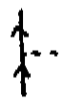
Physics 330 - Problem Set #9

Solutions

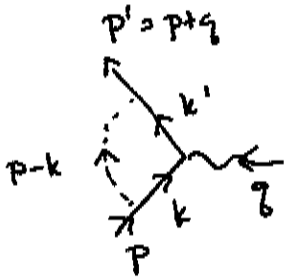
1.) a.) Using $\quad \Delta H = \lambda \phi \bar{\psi} \psi$



$$= \frac{i}{q^2 - M^2}$$



$$= -i\lambda$$



$$= (-ie)(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k+q)^2 - m^2} \frac{i}{k^2 - m^2} \frac{i}{(k-p)^2 - M^2}$$

$$\bar{u}(p') (\not{k} + \not{q} + m) \gamma^\mu (\not{k} + m) u(p)$$

$$= (-ie)(+i\lambda^2) \int dx\,dy\,dz\ \delta(x+y+z-1) \int \frac{d^4\ell}{(2\pi)^4} \frac{2}{[\ell^2 - \Delta]^3}$$

$$\cdot\ \bar{u}(p') (\not{k} + \not{q} + m) \gamma^\mu (\not{k} + m) u(p)$$

where

$$\ell^2 - \Delta = x((k+q)^2 - m^2) + y(k^2 - m^2) + z((k-p)^2 - M^2)$$

$$= k^2 + 2k\cdot(xq - zp) + xq^2 + zp^2 - (1-z)m^2 - zM^2$$

Let $\quad \ell = k + xq - zp$

$$\mathbb{k}^2 - \Delta = \mathbb{k}^2 - (xq - zp)^2 + xq^2 + zp^2 - (1-z)m^2 - zM^2$$

$$= \mathbb{k}^2 + x\underbrace{(1-x)}_{y+z} q^2 + 2xz\, p\cdot q + z\underbrace{(1-z)}_{x+y} p^2 - (1-z)m^2 - zM^2$$

$$= \mathbb{k}^2 + xyq^2 + xz(p+q)^2 + yzp^2 - (1-z)m^2 - zM^2$$

$$= \mathbb{k}^2 + xyq^2 + \underbrace{(xz+yz)}_{z(1-z)} m^2 - (1-z)m^2 - zM^2$$

$$= \mathbb{k}^2 + xyq^2 - (1-z)^2 m^2 - zM^2$$

recap:

$$\Delta = (1-z)^2 m^2 + zM^2 - xyq^2 \qquad (>0)$$

$$k = \mathbb{k} - xq + zp$$

$$k+q = \mathbb{k} + (1-x)q + zp$$

$$\bar{u}(p')(\not k + \not q + m)\, \gamma^\mu (\not k + m)\, u(p)$$

$$= \bar{u}(p') \left[[\not{\mathbb{k}} + (1-x)\not q + z\not p] + m \right] \gamma^\mu \left[(\not{\mathbb{k}} - x\not q + z\not p) + m \right] u(p)$$

drop terms linear in $\mathbb{k}$

$$= \bar{u}(p')\, \not{\mathbb{k}} \gamma^\mu \not{\mathbb{k}}\, u(p) + \bar{u}(p') [\overbrace{(1-x)}^{y+z}\not q + z\not p + m] \gamma^\mu \cdot [-xq + zp + m]\, u(p)$$

now [redacted] $z\not p\, u(p) = zm\, u(p)$

$$\bar{u}(p')\, z(\not p + \not q) = \bar{u}(p') \cdot zm$$

$$\bar{u}(p')\, \not{k}\gamma^\mu \not{k}\, u(p) = \tfrac{1}{4}k^2 \;\bar{u}(p')\,\gamma^\alpha\gamma^\mu\gamma_\alpha\, u(p)$$

$$= -\tfrac{1}{2}k^2\; \bar{u}(p')\,\gamma^\mu\, u(p)$$

contributes only to $F_1(q^2)$; drop this.

$$= \bar{u}(p')\,[y\not{q} + (1+z)m]\,\gamma^\mu\,[-x\not{q} + (1+z)m]\, u(p)$$

$$= \bar{u}(p')\,[\,(1+z)^2 m^2\,\gamma^\mu + (-x\,\gamma^\mu\not{q} + y\,\not{q}\gamma^\mu)(1+z)m$$
$$- xy\,\not{q}\gamma^\mu\not{q}\;]\; u(p)$$

now $(1+z)^2 m^2\gamma^\mu$ contributes only to F_1

$-xy\,\not{q}\gamma^\mu\not{q} = -xy\,[\,2q^\mu \underbrace{\not{q}}_{=0} - \gamma^\mu q^2\,]$ contributes only to F_1

$$-x\gamma^\mu\not{q} + y\not{q}\gamma^\mu = (-x)\;[\gamma^\mu, \not{q}] \qquad \text{since } \langle x\rangle = \langle y\rangle$$

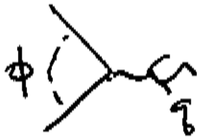
$$= (2ix)\;\tfrac{i}{2}[\gamma^\mu,\gamma^\nu]\,q_\nu$$

$$= (2ix)\cdot 2m\cdot \tfrac{1}{2m}\,\sigma^{\mu\nu}q_\nu$$

and
$\langle x\rangle = \langle y\rangle = \tfrac{1}{2}\langle x+y\rangle$
$= \tfrac{1}{2}(1-z)$

$$= 2m(1-z)\;\frac{i}{2m}\,\sigma^{\mu\nu}q_\nu$$

in all $\longrightarrow$



$$= (-ie)(+i\lambda^2) \int dx\,dy\,dz\, \delta(1-x-y-z)$$

$$\cdot \int \frac{d^4 k}{(2\pi)^4} \frac{2}{[k^2-\Delta]^3} \cdot \bar{u}(p')\, 2m(1-z) \cdot m(1+z) \left(\frac{i}{2m}\sigma^{\mu\nu} q_\nu\right) u(p)$$

$$+ \;(\text{terms contributing to } F_1(q^2))$$

$$F_2(q^2) = +i\lambda^2 \int dx\,dy\,dz\,\delta(1-x-y-z) \int \frac{d^4 k}{(2\pi)^4} \frac{2}{[k^2-\Delta]^3} \cdot 2m^2(1-z^2)$$

$$= +i\lambda^2 \int dx\,dy\,dz\,\delta(1-x-y-z)\; \frac{-i}{(4\pi)^2}\; \frac{2}{2\cdot\Delta}\; 2m^2(1-z)(1+z)$$

$$F_2(0) = +\frac{\lambda^2}{(4\pi)^2} \underbrace{\int dx\,dy\,dz\,\delta(1-x-y-z)}_{\int_0^1 dz \int_0^{1-z} dx} \frac{2m^2(1-z)(1+z)}{m^2(1-z)^2 + zM^2}$$

$$= +\frac{\lambda^2}{(4\pi)^2} \int_0^1 dz\, \frac{(1-z)^2(1+z)\cdot 2m^2}{m^2(1-z)^2 + zM^2}$$

since the correction to $(g-2)$ is small, we are interested in this integral for $M^2 \gg m^2$. It is tempting to approximate it as

$$\int_0^\infty dz\, \frac{(1-z)^2(1+z)}{zM^2}$$

but this is divergent at $z \to 0$. To control this, we need to keep the m^2 term in the denominator, but only in the divergent term, and only near $z \to 0$. That is, expand

$$(1-z)^2(1+z) = (1-z^2)(1-z) = 1 - z - z^2 + z^3$$

$$F_1(q^2=0) = +\frac{\lambda^2}{(4\pi)^2}\int_0^1 dz\; \frac{2m^2}{M^2}\left\{\frac{1}{(z+m^2/M^2)} + \frac{(-z-z^2+z^3)}{z}\right\}$$

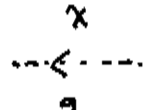
$$= +\frac{\lambda^2}{(4\pi)^2}\,\frac{2m^2}{M^2}\left\{\log\left(\frac{M^2}{m^2}\right) - 1 - \frac{1}{2} + \frac{1}{3}\right\}$$

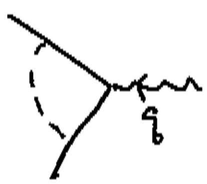
$$= +\frac{\lambda^2}{(4\pi)^2}\,\frac{2m^2}{M^2}\left(\log\frac{M^2}{m^2} - \frac{7}{6}\right) + O\left(\frac{m^4}{M^4}\right)$$

or

$$\boxed{\delta\left(\frac{g-2}{2}\right)_\phi = +\frac{\alpha_\lambda}{4\pi}\cdot\frac{2m^2}{M^2}\left(\log\frac{M^2}{m^2} - \frac{7}{6}\right)}$$

b.) Redo this for $\Delta H = i\lambda\,\chi\,\bar\psi\gamma^5\psi$

 $= \dfrac{i}{q^2 - M^2}$  $= +\lambda\gamma^5$



$$= (-ie)\,\lambda^2\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{(k+q)^2-m^2}\,\frac{i}{k^2-m^2}\,\frac{i}{(k-p)^2-M^2}$$

$$\cdot\ \bar u(p')\,\gamma^5\,(\not k + \not q + m)\,\gamma^\mu\,(\not k + m)\,\gamma^5\,u(p)$$

or, moving the γ^5's: $\gamma^5 \not{k} = -\not{k}\gamma^5$, $(\gamma^5)^2 = 1$

$$= (-ie)(+i\lambda^2)\int \frac{d^4k}{(2\pi)^4}\int dx\,dy\,dz\,\delta(1-x-y-z)\,\frac{2}{[\ell^2-\Delta]^3}$$

$$\cdot\ \bar{u}(p')(\not{k}+\not{q}-m)\,\gamma^\mu\,(\not{k}-m)\,u(p)$$

now, the denominator algebra is the same as before
the numerator changes to:

$$\bar{u}(p')(\not{k}+\not{q}-m)\,\gamma^\mu\,(\not{k}-m)\,u(p)$$

$$= \bar{u}(p')\,[\not{\ell}+\underbrace{(1-x)}_{y+z}\not{q}+z\not{p}-m]\,\gamma^\mu\,[\not{\ell}-x\not{q}+z\not{p}-m]\,u(p)$$

$$= \bar{u}(p')\,\not{\ell}\gamma^\mu\not{\ell}\,u(p) \quad\rightarrow\quad F_1 \text{ only}$$

$$+\ \bar{u}(p')\,([z(\not{p}+\not{q})-m]+y\not{q})\,\gamma^\mu\,(z\not{p}-m-x\not{q})\,u(p)$$

$$= (F_1) + \bar{u}(p')\,(-(1-z)m+y\not{q})\,\gamma^\mu\,(-(1-z)m-x\not{q})\,u(p)$$

$$= (F_1 \text{ terms}) - m(1-z)\cdot(-x)\ \bar{u}(p')\,[\gamma^\mu,\not{q}]\,u(p)$$

$$= (F_1 \text{ terms}) - m(1-z)\cdot\ 2m(1-z)\left(\bar{u}\,\frac{i\sigma^{\mu\nu}q_\nu}{2m}\,u\right)$$

so

$$F_2(q^2) = (+i\lambda^2)\int dx\,dy\,dz\,\delta(1-x-y-z)\int\frac{d^4\ell}{(2\pi)^4}\,\frac{2}{(\ell^2-\Delta)^3}$$

$$\cdot\ \left[-2m^2(1-z)^2\right]$$

$$F_2(0) = +i\lambda^2 \int dx\,dy\,dz\, \delta(1-x-y-z) \; \frac{-i}{(4\pi)^2} \; \frac{2}{2\cdot\Delta} \left(-2m^2(1-z)^2\right)$$

$$= -\frac{\lambda^2}{(4\pi)^2} \; 2m^2 \int_0^1 dz \int_0^{1-z} dx \; \frac{(1-z)^2}{(1-z)^2 m^2 + zM^2}$$

$$= -\frac{\lambda^2}{(4\pi)^2} \left(\frac{2m^2}{M^2}\right) \int_0^1 dz \; \frac{(1-z)^3}{z + (1-z)^2 m^2/M^2}$$

$M^2 \gg m^2$
$$\cong -\frac{\alpha_\lambda}{4\pi} \; \frac{2m^2}{M^2} \int_0^1 dz \left(\frac{1}{z + m^2/M^2} + \frac{-3z + 3z^2 - z^3}{z}\right)$$

$$= -\frac{\alpha_\lambda}{4\pi} \; \frac{2m^2}{M^2} \left[\lg M^2/m^2 - 3 + \frac{3}{2} - \frac{1}{3}\right]$$

$$F_2(0) = -\frac{\alpha_\lambda}{4\pi} \; \frac{2m^2}{M^2} \left[\lg M^2/m^2 - \frac{11}{6}\right]$$

so

$$\boxed{\begin{aligned} \delta\left(\frac{(g-2)}{2}\right)_\phi &= +\frac{\alpha_\lambda}{2\pi} \frac{m^2}{M^2}\left[\lg M^2/m^2 - 7/6\right] \\ \delta\left(\frac{g-2}{2}\right)_\chi &= -\frac{\alpha_\lambda}{2\pi} \frac{m^2}{M^2}\left[\lg M^2/m^2 - \frac{11}{6}\right] \end{aligned}} \; (*)$$

(*) thanks to Zdangbek Atpishev.

c.) Note that the scalar gives a positive contribution to $(g-2)$. The pseudoscalar gives a negative contribution.

So, to obtain $$\Delta\left(\frac{g-2}{2}\right) = 2\times10^{-9}$$

we need to take the scalar model.

for $\alpha_\lambda = \frac{1}{100}$ we obtain this value for $\frac{m}{M} = 2.9\times10^{-4}$

$$\text{i.e.}\quad m_\mu = 106\ \text{MeV} \longrightarrow M = 370\ \text{GeV}$$

more generally $\log M^2/m^2 - \frac{7}{6} \approx 15$

$$\frac{m}{M} = 3\times10^{-5}\frac{1}{\sqrt{\alpha_\lambda}} \quad \text{or} \quad M \approx 3700\ \text{GeV}\cdot\sqrt{\alpha_\lambda}$$

d.) For the electron, the effect is smaller by

$$\left(\frac{m_e}{m_\mu}\right)^2 = \left(\frac{0.51\ \text{MeV}}{106\ \text{MeV}}\right)^2 = 2\times10^{-5}$$

so if the effect is there for the μ, it is invisible for the e^-.