Physics 330 – Problem Set # 8
Solutions

1.) a.) In the CM frame $P^\mu = (P, \vec{0})$

$$E_1 + E_2 + E_3 = P \qquad s = P^2$$

then

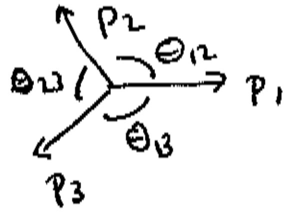
$$x_1 + x_2 + x_3 = \frac{2}{\sqrt{s}}(E_1 + E_2 + E_3) = \frac{2}{P}\cdot P = 2$$

b.)
$$m_{12}^2 = (p_1+p_2)^2 = (P-p_3)^2 = P^2 - 2P\cdot p_3 + m_3^2$$
$$= s - 2P\cdot E_3 + m_3^2$$
$$= s - 2\sqrt{s}\cdot\frac{\sqrt{s}}{2}x_3 + m_3^2$$

so
$$m_{12}^2 = s(1-x_3) + m_3^2$$

similarly
$$m_{23}^2 = s(1-x_1) + m_1^2$$
$$m_{31}^2 = s(1-x_2) + m_2^2$$

c.) In the CM frame $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$; then $\vec{p}_3$ lies in the plane determined by $\vec{p}_1, \vec{p}_2$.



$\cos\theta_{12}$ is given by

$$m_{12}^2 = m_1^2 + m_2^2 + 2p_i p_2$$

$$= m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\theta_{12}$$

Let $\quad x_i = \frac{2}{P}E_i \qquad y_i = \frac{2}{P}p_i = \frac{2}{P}\left[E_i^2 - m_i^2\right]^{\frac{1}{2}} = \left(x_i^2 - \frac{4}{s}m_i^2\right)^{\frac{1}{2}}$

$$m_{12}^2 = s(1-x_3) + m_3^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\theta_{12}$$

$$= m_1^2 + m_2^2 + s\left(x_1x_2 - y_1y_2\cos\theta_{12}\right)$$

so

$$\cos\theta_{12} = \frac{x_1x_2 + x_3 - 1 + (m_1^2 + m_2^2 - m_3^2)/s}{y_1y_2}$$

similarly

$$\cos\theta_{23} = \frac{x_2x_3 + x_1 - 1 + (m_2^2 + m_3^2 - m_1^2)/s}{y_2y_3}$$

$$\cos\theta_{31} = \frac{x_3x_1 + x_2 - 1 + (m_3^2 + m_1^2 - m_2^2)/s}{y_3y_1}$$

d.) Now choose coordinates for the event plane so ~~that~~ the $\hat{3}$ axis points along $\vec{p}_1$ and $\vec{p}_1, \vec{p}_2, \vec{p}_3$ lie in the $\hat{3}$-$\hat{1}$ plane.

To reorient to a general orientation, we integrate over $d\Omega_1$ and over the azimuthal angle of $\vec{p}_2$:

$$\int d\Omega_1\, d\varphi_2 = 4\pi \cdot 2\pi = 8\pi^2$$

then

$$\int d\Pi_3 = \int \frac{d^3p_1\, d^3p_2\, d^3p_3}{(2\pi)^9\; 2E_1 2E_2 2E_3} (2\pi)^4 \delta(P - p_1 - p_2 - p_3)$$

$$= \frac{1}{(2\pi)^6} \int \frac{d^3p_1\, d^3p_2}{2E_1 2E_2 2E_3}\; 2\pi\, \delta(P - E_1 - E_2 - E_3)$$

$$= \frac{1}{(2\pi)^6} \int \frac{dp_1\, p_1^2\, d\Omega_1\; dp_2\, p_2^2\, d\cos\theta_{12}\, d\varphi_2}{2E_1\, 2E_2\, 2E_3}\; 2\pi\, \delta(P - E_1 - E_2 - E_3)$$

$$= \frac{8\pi^2}{(2\pi)^6} \int \frac{dp_1\, p_1^2\; dp_2\, p_2^2\, d\cos\theta_{12}}{2E_1\, 2E_2\, 2E_3}\, 2\pi\, \delta(P - E_1 - E_2 - E_3)$$

now E_3 is determined by $\vec{p}_3 = -\vec{p}_1 - \vec{p}_2$

$$E_3 = \left[p_1^2 + p_1 p_2 \cos\theta + p_2^2 + m_3^2\right]^{\frac{1}{2}}$$

we can eliminate $d\cos\theta_{12}$ by integrating over the energy δ-function

$$\int d\cos\theta_{12}\; \delta(P - E_1 - E_2 - E_3) = \frac{1}{\left(\frac{p_1 p_2}{E_3}\right)}$$

in the region where $-1 \le \cos\theta_{12} \le 1$.

Finally, use $\quad dp_1\, p_1 = \frac{1}{2} dp_1^2 = \frac{1}{2} dE_1^2 = dE_1\, E_1 \quad$ similarly for E_2

$$\int d\Pi_3 = \frac{8\pi^2}{(2\pi)^6}\, 2\pi\, \frac{1}{8} \int \frac{dE_1\, E_1\, p_1\;\; dE_2\, E_2\, p_2}{E_1 E_2 E_3 \quad \left(\frac{p_1 p_2}{E_3}\right)}$$

$$= \frac{1}{32\pi^3} \int dE_1\, dE_2 \quad = \quad \frac{s}{128\pi^3} \int dx_1\, dx_2$$

e.) Using the result of part (b), this can also be written

$$dm_{12}^2 = -s\, dx_3 \quad \text{etc.}$$

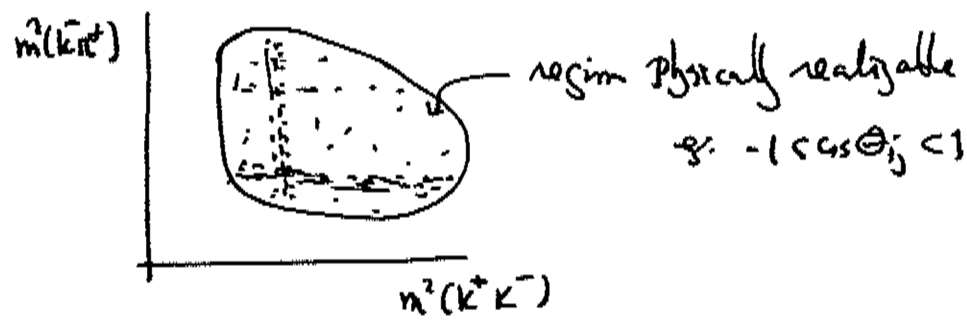
$$\int d\Pi_3 = \frac{1}{128\pi^3 s} \int dm_{23}^2\, dm_{13}^2$$

so

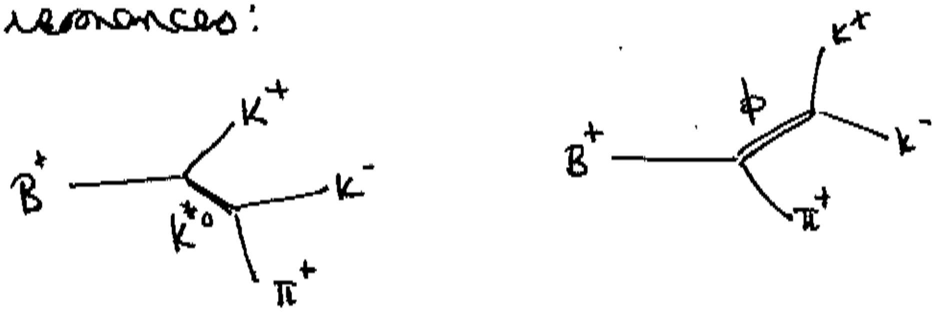
$$\frac{d^2\Pi_3}{dm_{12}^2\, dm_{13}^2} \leftarrow \text{any two } m_{ij}^2\text{'s here}$$

$$= \underline{(\text{const})}$$

consider e.g. $B^+ \to K^+K^-\pi^+$. Points accumulate along lines in the $m^2(K^+K^-) - m^2(K^-\pi^+)$ plane:

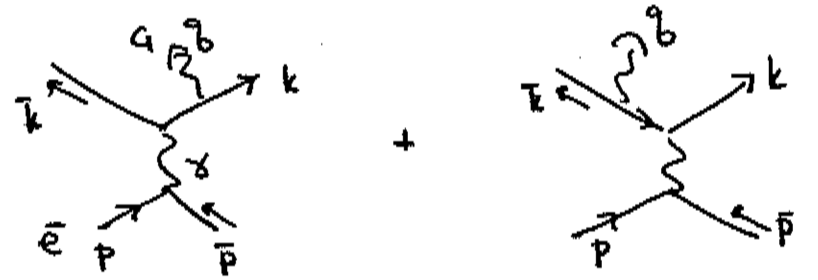


This accumulation cannot be due to phase space, they must be resonances:



2.) a.) The G_μ vertex is  $\mu = +ig\gamma^\mu$

The process of G emission is given by



$$= (-ie)(+iQ_f e)(+ig) \quad \bar{v}(\bar{p})\gamma^\mu u(p) \frac{-i}{(p+\bar{p})^2}$$

$$\cdot \quad \bar{u}(k)\left[\gamma\cdot\varepsilon^*(q) \frac{i(\not{k}+\not{q})}{(k+q)^2}\gamma_\mu + \gamma_\mu \frac{i[-(\not{\bar{k}}+\not{q})]}{(\bar{k}+q)^2}\gamma\cdot\varepsilon^*(q) \right] v(\bar{k})$$

$$= iQ_f e^2 g \frac{1}{s} \quad \bar{v}(\bar{p})\gamma^\mu u(p) \cdot \bar{u}(k)\left[\frac{\gamma\cdot\varepsilon^*(q)(\not{k}+\not{q})\gamma_\mu}{2k\cdot q} - \frac{\gamma_\mu(\not{\bar{k}}+\not{q})\gamma\cdot\varepsilon^*(q)}{2\bar{k}\cdot q}\right] v(\bar{k})$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = Q_f^2\frac{e^4 g^2}{4}\frac{1}{s^2} \quad tr[\not{\bar{p}}\gamma^\mu \not{p}\gamma^\lambda] \quad (-g_{\nu\sigma})$$

$$\cdot\, tr\Big[\not{k}\Big\{\frac{\gamma_\nu(\not{k}+\not{q})\gamma_\mu}{2k\cdot q} - \gamma_\mu\frac{(\not{\bar{k}}+\not{q})\gamma_\nu}{2\bar{k}\cdot q}\Big\}$$

$$\cdot\,\not{\bar{k}}\Big\{\frac{\gamma_\lambda(\not{k}+\not{q})\gamma_\sigma}{2k\cdot q} - \gamma_\sigma\frac{(\not{\bar{k}}+\not{q})\gamma_\lambda}{2\bar{k}\cdot q}\Big\}\Big]$$

$\times$ 3 for color

b.) The electron-positron trace is

$$\mathrm{tr}(\bar{\not p}\,\gamma^\mu \not p\,\gamma^\lambda) = 4\,[\bar{p}^\mu p^\lambda + p^\mu \bar{p}^\lambda - p\cdot\bar{p}\, g^{\mu\lambda}]$$

Instead of averaging the final state over orientations, average this over orientations

$$p = (E, E\hat{n}) \qquad \bar{p} = (E, -E\hat{n}) \qquad p\cdot\bar{p} = 2E^2$$

$$(\text{atre}) = 4 \begin{array}{c|cc} {}_{\mu=}\,{}^{\lambda=} & 0 & j \\ 0 & \multicolumn{2}{c}{\left(\begin{array}{c|c} E^2+E^2-2E^2 & E\cdot E\,\hat{n}^j + E\cdot E(-\hat{n}) \\ \hline E(-E\hat{n}) + E\cdot E\hat{n} & -E^2\hat{n}^i\hat{n}^j - E^2\hat{n}^i\hat{n}^j + 2E^2\delta^{ij} \end{array}\right)} \\ i & & \end{array}$$

$$= 4 \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 2E^2(\delta^{ij} - \hat{n}^i\hat{n}^j) \end{array}\right)$$

average over orientation of $\hat{n}$

$$\langle \text{atre} \rangle = 4\cdot 2E^2 \cdot \left(\begin{array}{c|c} 0 & \\ \hline & \tfrac{2}{3}\delta^{ij} \end{array}\right)$$

$$= 4\cdot\left[\frac{4E^2}{3}\left(\begin{array}{c|c} -1 & \\ \hline & \delta^{ij} \end{array}\right) + \frac{4E^3}{3}\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right)\right]$$

$$= 4\left\{P^2\,\frac{1}{3}\left(-g^{\mu\nu}\right) + \frac{1}{3}P^\mu P^\nu\right\}$$

where $\quad P = p + \bar{p} = (2E, \vec{0})$

again $$\langle \mathrm{tr}(\bar{p}\!\!\!/ \gamma^\mu p\!\!\!/ \gamma^\nu)\rangle = -\frac{4}{3}\left(P^2 g^{\mu\nu} - P^\mu P^\nu\right)$$

We can check this by the following properties:

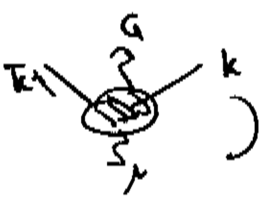
(1) this object must be a function of P^μ only

(2) dotting in P^μ gives zero, since
$$P^\mu\, \bar{v}(\bar p)\gamma_\mu u(p) = \bar v(\bar p)(\bar{p}\!\!\!/ + p\!\!\!/)\,u(p) = 0$$

(3) dotting w. $g_{\mu\lambda}$ gives
$$g_{\mu\lambda}\,\mathrm{tr}(\bar{p}\!\!\!/\gamma^\mu p\!\!\!/\gamma^\lambda) = \mathrm{tr}(\bar{p}\!\!\!/\gamma^\mu p\!\!\!/ \gamma_\mu) = \mathrm{tr}(\bar{p}\!\!\!/(-2p\!\!\!/)) = -8p\bar p = -4(2p\cdot\bar p) = -4P^2$$

to satisfy (2) we must have
$$\langle \mathrm{tr}(\bar{p}\!\!\!/ \gamma^\mu p\!\!\!/ \gamma^\nu)\rangle = A\left(P^2 g^{\mu\nu} - P^\mu P^\nu\right)$$
and (3) gives $A = -\frac{4}{3}$.

c.) Notice that P^μ dotted into the second trace gives zero:

$P_\mu \cdot \Big($ 

$\Big) = 0$

more explicitly, dotting with P_μ gives

$$= \bar u(k)\left[\frac{\gamma\cdot\epsilon^*(q)\,(k\!\!\!/+q\!\!\!/)\,P\!\!\!\!/}{2k\cdot q} - P\!\!\!\!/\,\frac{(\bar{k}\!\!\!/+q\!\!\!/)\,\gamma\cdot\epsilon^*(q)}{\bar k\cdot q}\right] v(\bar k)$$

then $(\not{k}+\not{q})\not{\bar{k}}\,v(\bar{k}) = (\not{k}+\not{q})(\not{k}+\not{q}+\not{\bar{k}})\,v(k)$

now $\not{\bar{k}}\,v(\bar{k}) = 0 \qquad (\not{k}+\not{q})^2 = (k+q)^2 = 2k\cdot q$

$$\bar{u}(k)\not{k}(\not{\bar{k}}+\not{q}) = \bar{u}(k)(\cancel{\not{k}}+\not{\bar{k}}+\not{q})(\not{\bar{k}}+\not{q}) = \bar{u}(k)(\not{\bar{k}}+\not{q})^2$$
$$= \bar{u}(2\bar{k}\cdot q)$$

so the above is

$$\bar{u}(k)\left\{ \gamma\cdot\varepsilon^*(q)\,\frac{2k\cdot q}{2k\cdot q} - \frac{2\bar{k}\cdot q}{2\bar{k}\cdot q}\,\gamma\cdot\varepsilon^*(q)\right\} v(\bar{k})$$
$$= 0$$

now $\frac{1}{4}\sum|\mathcal{M}|^2$ reduces to:

$$\frac{1}{4}\sum|\mathcal{M}|^2 = 3\cdot\frac{Q_f^2 e^4 g^2}{4}\,\frac{1}{s^2}\,(-1)\cdot\left(-\frac{4}{3}s\right)$$
$$\cdot\,\mathrm{tr}\left[\not{k}\left\{\frac{\gamma_\nu(\not{k}+\not{q})\gamma_\mu}{2k\cdot q} - \frac{\gamma_\mu(\not{\bar{k}}+\not{q})\gamma_\nu}{2\bar{k}\cdot q}\right\}\right.$$
$$\left.\not{\bar{k}}\left\{\frac{\gamma_\mu(\not{k}+\not{q})\gamma_\nu}{2k\cdot q} - \frac{\gamma_\nu(\not{\bar{k}}+\not{q})\gamma_\mu}{2\bar{k}\cdot q}\right\}\right]$$

Now we have to bite the bullet and do this trace.

$$\mathrm{tr}\,[\,] = \frac{\mathrm{I}}{(2k\cdot q)^2} - \frac{\mathrm{II}}{2k\cdot q\,2\bar{k}\cdot q} - \frac{\mathrm{III}}{2\bar{k}\cdot q\,2k\cdot q} + \frac{\mathrm{IV}}{(2\bar{k}\cdot q)^2}$$

$$\mathrm{I} = \mathrm{tr}\ \not{k}\ \gamma_\nu (\not{k}+\not{q}) \gamma_\mu\ \bar{\not{k}}\ \gamma^\mu (\not{k}+\not{q}) \gamma^\nu$$

$$= \mathrm{tr}\ \gamma^\nu \not{k}\ \gamma_\nu (\not{k}+\not{q})\ \gamma^\mu\ \bar{\not{k}}\ \gamma_\mu (\not{k}+\not{q})$$

$$= \mathrm{tr}\ (-2\not{k})\ (\not{k}+\not{q})\ (-2\bar{\not{k}})\ (\not{k}+\not{q})$$

$$= (-2)^2 \cdot 4\ [\,k\cdot(k+q)\ \bar{k}\cdot(k+q)\cdot 2 - k\cdot\bar{k}\ (k+q)^2\,]$$

$$= 16\ [\,2k\cdot q\,(\cancel{\bar{k}\cdot k} + \bar{k}\cdot q) - \cancel{k\cdot\bar{k}\cdot 2k\cdot q}\,]$$

$$= 32\ k\cdot q\ \bar{k}\cdot q$$

$$\mathrm{IV} = \mathrm{tr}\ \not{k}\ \gamma_\mu (\bar{\not{k}}+\not{q}) \gamma_\nu\ \bar{\not{k}}\ \gamma^\nu (\bar{\not{k}}+\not{q}) \gamma^\mu$$

$$= \mathrm{tr}\ (-2)^2\ \not{k}\ (\bar{\not{k}}+\not{q})\ \bar{\not{k}}\ (\bar{\not{k}}+\not{q})$$

$$= 16\cdot \{\, 2k\cdot(\bar{k}+q)\ \bar{k}\cdot(\bar{k}+q) - k\cdot\bar{k}\ (\bar{k}+q)^2 \,\}$$

$$= 16\ \{\, (\cancel{2\bar{k}\cdot k} + 2k\cdot q)\ \bar{k}\cdot q - \cancel{k\cdot\bar{k}\cdot 2\bar{k}\cdot q} \,\}$$

$$= 32\ k\cdot q\ \bar{k}\cdot q$$

$$\mathrm{II} = \mathrm{tr}\,[\not{k}\ \gamma_\nu (\not{k}+\not{q}) \gamma_\mu\ \bar{\not{k}}\ \gamma^\nu (\bar{\not{k}}+\not{q}) \gamma^\mu]$$

$$= \mathrm{tr}\ \not{k}\ \gamma_\nu (\not{k}+\not{q})\ (-2)(\bar{\not{k}}+\not{q})\ \gamma^\nu\ \bar{\not{k}}$$

$$= \mathrm{tr}\ (-2)\cdot[\,\not{k}\cdot 4\,(k+q)\cdot(\bar{k}+q)\ \bar{\not{k}}\,]$$

$$= -8\cdot 4\cdot k\cdot\bar{k}\ \cdot(k\cdot\bar{k} + q\cdot\bar{k} + q\cdot k)$$

now $\quad 2(k\cdot\bar{k} + q\cdot\bar{k} + q\cdot k) = (k+\bar{k}+q)^2 = S$

$$= -16\ k\cdot\bar{k}\cdot S$$

$$\mathrm{III} = \mathrm{tr}\left[\not{k}\, \gamma_\mu (\not{\bar{k}}+\not{q})\, \gamma_\nu\, \not{\bar{k}}\, \gamma^\mu (\not{k}+\not{q})\, \gamma^\nu\right]$$

$$= \mathrm{tr}\ \not{k}\ (-2)\ \not{\bar{k}}\, \gamma_\nu (\not{\bar{k}}+\not{q})(\not{k}+\not{q})\, \gamma^\nu$$

$$= (-2)\cdot 4\ \mathrm{tr}\ \not{k}\ \not{\bar{k}}\ (\bar{k}+q)\cdot(k+q)$$

$$= -16\ k\cdot\bar{k}\cdot s \qquad \text{as in } \mathrm{II}$$

$$\mathrm{tr}[\ ] = 8\cdot\left\{ \frac{(2k\cdot q)(2\bar{k}\cdot q)}{(2k\cdot q)^2} + 2\,\frac{2k\cdot\bar{k}\cdot s}{2k\cdot q\ 2\bar{k}\cdot q} + \frac{2k\cdot q\ 2\bar{k}\cdot q}{(2\bar{k}\cdot q)^2} \right\}$$

$$= 8\left\{ \frac{2\bar{k}\cdot q}{2k\cdot q} + \frac{2k\cdot q}{2\bar{k}\cdot q} + 2\,\frac{2k\cdot\bar{k}\, s}{2k\cdot q\ 2\bar{k}\cdot q} \right\}$$

now

$$2\bar{k}\cdot q = (q+\bar{k})^2 = (P-k)^2 = s\cdot(1-x_k)$$
$$2k\cdot q = (q+k)^2 = (P-\bar{k})^2 = s(1-x_{\bar{k}})$$
$$2k\cdot\bar{k} = (k+\bar{k})^2 = (P-q)^2 = s(1-x_q) = s(x_k+x_{\bar{k}}-1)$$

set

$$x_k = \frac{2}{P}k^0 \qquad x_{\bar{k}} = \frac{2}{P}\bar{k}^0 \qquad x_q = \frac{2}{P}q^0$$

$$x_k + x_{\bar{k}} + x_q = 2$$

$$\mathrm{tr}[\ ] = 8\left\{ \frac{(1-x_k)}{(1-x_{\bar{k}})} + \frac{(1-x_{\bar{k}})}{(1-x_k)} + 2\,\frac{(x_k+x_{\bar{k}}-1)}{(1-x_k)(1-x_{\bar{k}})} \right\}$$

$$= 8\left(\frac{(1-x_k)^2 + (1-x_{\bar{k}})^2 + 2x_k + 2x_{\bar{k}} - 2}{(1-x_k)(1-x_{\bar{k}})} \right)$$

$$= 8\,\frac{x_k^2 + x_{\bar{k}}^2}{(1-x_k)(1-x_{\bar{k}})}$$

so!

$$\frac{1}{4}\Sigma|\mathcal{M}|^2 = 3 \cdot \frac{Q_f^2 e^4 g^2}{3s} \cdot 8 \cdot \frac{x_k^2 + x_{\bar{k}}^2}{(1-x_k)(1-x_{\bar{k}})}$$

$$\int d\sigma = \frac{1}{2s} \cdot \int d\Pi_3 \, \frac{1}{4}\Sigma|\mathcal{M}|^2$$

$$= \frac{1}{2s} \cdot \frac{s}{128\pi^3} \int dx_k dx_{\bar{k}} \frac{3Q_f^2 e^4 g^2}{3s} \cdot 8 \cdot \frac{x_k^2 + x_{\bar{k}}^2}{(1-x_k)(1-x_{\bar{k}})}$$

$$= \int dx_k dx_{\bar{k}} \frac{3Q_f^2 e^4 g^2}{3 \cdot 32\pi^3 s} \frac{x_k^2 + x_{\bar{k}}^2}{(1-x_k)(1-x_{\bar{k}})}$$

$$= \int dx_k dx_{\bar{k}} \frac{2 \cdot 3}{3s} Q_f^2 \, \alpha^2 \alpha_g \frac{x_k^2 + x_{\bar{k}}^2}{(1-x_k)(1-x_{\bar{k}})}$$

$$\frac{d^2\sigma}{dx_k dx_{\bar{k}}} = \frac{4\pi\alpha^2}{3s} \cdot 3Q_f^2 \cdot \frac{\alpha_g}{2\pi} \frac{x_k^2 + x_{\bar{k}}^2}{(1-x_k)(1-x_{\bar{k}})}$$