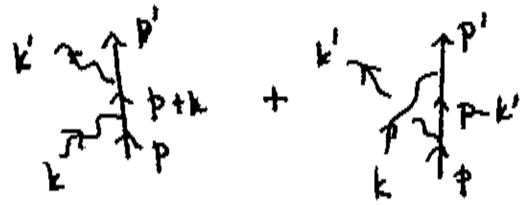
# Physics 330 - Problem Set #7
## Solutions

1.) 

$$i\mathcal{M} = (-ie)^2\, \bar{u}(p') \Big\{ \gamma^\mu \varepsilon^*_\mu(k') \frac{i(\not p + \not k + m)}{(p+k)^2 - m^2} \gamma^\nu \varepsilon_\nu(k)$$

$$+ \gamma^\nu \varepsilon_\nu(k) \frac{i(\not p - \not k' + m)}{(p-k')^2 - m^2} \gamma^\mu \varepsilon^*_\mu(k') \Big\} u(p)$$

now $$(\not p + m)\gamma^\nu u(p) = [2p^\nu - \gamma^\nu(\not p - m)]\, u(p)$$
$$= 2p^\nu u(p)$$

$$(p+k)^2 - m^2 = m^2 + 2p\cdot k + k^2 - m^2 = 2p\cdot k$$
$$(p-k')^2 - m^2 = m^2 - 2p\cdot k' + k'^2 - m^2 = -2p\cdot k'$$

kinematic relations are:

$$s = (p+k)^2 = (p'+k')^2 \Rightarrow \cancel{m^2} + 2p\cdot k = \cancel{m^2} + 2p'\cdot k'$$
$$t = (p-p')^2 = (k-k')^2 \Rightarrow 2m^2 - 2p\cdot p' = -2kk'$$
$$u = (p-k')^2 = (k-p')^2 \Rightarrow m^2 - 2p\cdot k' = m^2 - 2k\cdot p$$

so $\quad p\cdot k = p'\cdot k' \qquad p\cdot k' = p'\cdot k \qquad p\cdot p' = m^2 + k\cdot k'$

$$p\cdot(k' + p' - k) = p^2 = m^2 \Rightarrow p\cdot p' = m^2 + p\cdot k - p\cdot k'$$

I will try to systematically eliminate $p'\cdot k'$, $p'\cdot k$, $k\cdot k'$ in favor of $p\cdot k$, $p\cdot k'$, $p\cdot p'$

$$i\mathcal{M} = -ie^2 \varepsilon_\mu^{*}(k')\varepsilon_\nu(k) \cdot \bar{u}(p')\left[\frac{2\gamma^\mu p^\nu + \gamma^\mu \not{k}\gamma^\nu}{2p\cdot k} - \frac{2\gamma^\nu p^\mu - \gamma^\nu \not{k}'\gamma^\mu}{2p\cdot k'}\right]u(p)$$

then

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4}\,\mathrm{tr}\left[(\not{p}'+m)\left\{\frac{2\gamma^\mu p^\nu + \gamma^\mu\not{k}\gamma^\nu}{2p\cdot k} + \frac{-2\gamma^\nu p^\mu + \gamma^\nu\not{k}'\gamma^\mu}{2p\cdot k'}\right\} \cdot (\not{p}+m)\left\{\frac{2\gamma_\mu p_\nu + \gamma_\nu\not{k}\gamma_\mu}{2p\cdot k} + \frac{-2\gamma_\nu p_\mu + \gamma_\mu\not{k}'\gamma_\nu}{2p\cdot k'}\right\}\right]$$

$$= \frac{e^4}{4}\left[\frac{\mathrm{I}}{(2p\cdot k)^2} + \frac{\mathrm{II}}{2p\cdot k\, 2p\cdot k'} + \frac{\mathrm{III}}{2p\cdot k'\, 2p\cdot k} + \frac{\mathrm{IV}}{(2p\cdot k')^2}\right]$$

now compute I, II, III, IV in turn.

$$\begin{aligned}
\mathrm{I} &= \mathrm{tr}\,(\not{p}'+m)(2\gamma^\mu p^\nu + \gamma^\mu\not{k}\gamma^\nu)(\not{p}+m)(2\gamma_\mu p_\nu + \gamma_\nu\not{k}\gamma_\mu) \\
&= \mathrm{tr}\,(\not{p}'+m)\,4\gamma^\mu(\not{p}+m)\gamma_\mu\, p^2 \\
&\quad + \mathrm{tr}\,(\not{p}'+m)\cdot 2\gamma^\mu(\not{p}+m)\not{p}\not{k}\gamma_\mu \\
&\quad + \mathrm{tr}\,(\not{p}'+m)\gamma^\mu\not{k}\not{p}(\not{p}+m)\gamma_\mu\cdot 2 \\
&\quad + \mathrm{tr}\,(\not{p}'+m)\gamma^\mu\not{k}\gamma^\nu(\not{p}+m)\gamma_\nu\not{k}\gamma_\mu
\end{aligned}$$

Here

$$\text{line } 1 = 4m^2 \operatorname{tr} (\not{p}'+m)\gamma^\mu(\not{p}+m)\gamma_\mu$$
$$= 4m^2 \operatorname{tr} (\not{p}'+m)(-2\not{p}+4m)$$
$$= 4\cdot(-2)\cdot 4\, m^2\, p'\cdot p + 4\cdot 4\cdot 4\, m^4$$
$$= -32\, p'\cdot p\, m^2 + 64 m^4$$

$$\text{line } 2 = 2 \operatorname{tr} (\not{p}'+m)\gamma^\mu(\not{p}+m)\not{p}\not{k}\gamma_\mu$$
$$= 2 \operatorname{tr} (\not{p}'+m)\gamma^\mu(m^2+m\not{p})\not{k}\gamma_\mu$$
$$= 2 \operatorname{tr} (\not{p}'+m)(-2\not{k}m^2 + 4\, p\cdot k\, m)$$
$$= -4\cdot 4\, m^2(p'\cdot k) + 2\cdot 4\cdot 4\, m^2\, p\cdot k$$
$$= -16 m^2 (p\cdot k') + 32 m^2\, p\cdot k$$

$$\text{line } 3 = 2 \operatorname{tr} (\not{p}'+m)\gamma^\mu\not{k}\not{p}(\not{p}+m)\gamma_\mu$$
$$= 2 \operatorname{tr} (\not{p}'+m)\gamma^\mu\not{k}(m^2+m\not{p})\gamma_\mu$$
$$= 2 \operatorname{tr} (\not{p}'+m)(-2\not{k}m^2 + 4\, p\cdot k\, m)$$
$$= -4\cdot 4(p'\cdot k)m^2 + 2\cdot 4\cdot 4\, m^2\, p\cdot k$$
$$= -16\, p\cdot k'\, m^2 + 32 m^2\, p\cdot k$$

I have used my trics : $\operatorname{tr}[\text{odd}\ \#\ \text{of}\ \gamma\text{'s}] = 0$

line 4 $= \mathrm{tr}\,(\not{p}'+m)\,\gamma^\mu \not{k}\gamma^\nu (\not{p}+m)\gamma_\nu \not{k}\gamma_\mu$

$= \mathrm{tr}\,\gamma_\mu (\not{p}'+m)\,\gamma^\mu \not{k}\, \gamma^\nu (\not{p}+m)\,\gamma_\nu \not{k}$

$= \mathrm{tr}\,(-2\not{p}'+4m)\not{k}\,(-2\not{p}+4m)\not{k}$

$= 4\cdot(-2)^2[\,p'\!\cdot k\; p\cdot k\cdot 2 - p'\!\cdot p\,\underbrace{k^2}_{=0}\,] + 4\cdot 4\cdot 4 m^2 \underbrace{k^2}_{=0}$

$= 32\, p\cdot k'\, p\cdot k$

in all

$$I = -32\,p'\!\cdot p\, m^2 + 64m^4 + 2(-16m^2\, p\cdot k' + 32m^2\, p\cdot k)$$
$$+\; 32\, p\cdot k\, p\cdot k'$$

$$= -32m^2(m^2 + p\cdot k - \cancel{p\cdot k'}) + 64m^2 - 32\cancel{p\cdot k'}m^2 + 64m^2\, p\cdot k$$
$$+\; 32\, p\cdot k\; p\cdot k'$$

$$= 32m^4 + 32m^2\, p\cdot k + 32\, p\cdot k\, p\cdot k'$$

In eq. (5.84) of P+S it is claimed:

$$I = 16\,(2m^4 + m^2(s-m^2) - \tfrac{1}{2}(s-m^2)(u-m^2))$$
$$= 32m^4 + 16m^2\cdot 2p\cdot k - \tfrac{1}{2}\cdot 16\cdot(2p\cdot k)(-2p\cdot k')$$
$$= 32m^4 + 32m^2\, p\cdot k + 32\, p\cdot k\; p k' \quad \checkmark$$

$$\text{IV} = \mathrm{tr}\,(\not{p}'+m)[2\gamma^\nu p^\mu - \gamma^\nu \not{k}' \gamma^\mu](\not{p}+m)[2\gamma_\nu p_\mu - \gamma_\mu \not{k}' \gamma_\nu]$$

which is I with $(\mu \leftrightarrow \nu)$, $k \leftrightarrow -k'$
so we can write the answer.

$$\text{IV} = 32[m^4 - m^2\, p\cdot k' + p\cdot k\; p\cdot k']$$

$$\text{II} = \mathrm{tr}\,(\not{p}'+m)(2\gamma^\mu p^\nu + \gamma^\mu \not{k} \gamma^\nu)(\not{p}+m)(-2\gamma_\nu p_\mu + \gamma_\mu \not{k}' \gamma_\nu)$$

$$\text{III} = \mathrm{tr}\,(\not{p}'+m)(-2\gamma_\nu p_\mu + \gamma_\nu \not{k}' \gamma_\mu)(\not{p}+m)(2\gamma^\mu p^\nu + \gamma^\nu \not{k} \gamma^\mu)$$

$$= \mathrm{tr}\,(-2\gamma_\nu p_\mu + \gamma_\nu \not{k}' \gamma_\mu)(\not{p}+m)(-2\gamma_\nu p_\mu + \gamma_\nu \not{k}' \gamma_\mu)(\not{p}'+m)$$

= II w. order of $\gamma$s in the trace reversed.

= II by eq. (5.7)

$$\text{II} = \mathrm{tr}(\not{p}'+m)\, 2\gamma^\mu p^\nu\, (\not{p}+m)(-2\gamma_\nu p_\mu)$$
$$+ \mathrm{tr}(\not{p}'+m)\, 2\gamma^\mu p^\nu\, (\not{p}+m)(\gamma_\mu \not{k}' \gamma_\nu)$$
$$+ \mathrm{tr}(\not{p}'+m)\, \gamma^\mu \not{k} \gamma^\nu\, (\not{p}+m)(-2\gamma_\nu p_\mu)$$
$$+ \mathrm{tr}(\not{p}'+m)\gamma^\mu \not{k} \gamma^\nu (\not{p}+m)\, \gamma_\mu \not{k}' \gamma_\nu$$

line 1 $= -4\, \mathrm{tr}(\not{p}'+m)\, \not{p}\, (\not{p}+m)\, \not{p}$

$= -4\, \mathrm{tr}\, (\not{p}'+m)\, (m^2 \not{p} + m^3)$

$= -4 \cdot 4\, (p\cdot p'\, m^2 + m^4)$

$= -16\, (p\cdot p'\, m^2 + m^4)$

line 2 $= 2\,\mathrm{tr}(\not{p}'+m)\gamma^\mu(\not{p}+m)\gamma_\mu \not{k}'\not{p}$

$= 2\,\mathrm{tr}(\not{p}'+m)(-2\not{p}+4m)\not{k}'\not{p}$

$= -4\cdot4\,[p'\cdot p\; k'\cdot p\cdot 2 - p\cdot k' m^2] + 2\cdot4\cdot4m^2\, k'\cdot p$

$= -32\, p'\cdot p\; k'\cdot p + 16m^2\, p\cdot k + 32m^2\, p\cdot k'$

line 3 $= -2\,\mathrm{tr}(\not{p}'+m)\not{p}\not{k}\gamma^\nu(\not{p}+m)\gamma_\nu$

$= -2\,\mathrm{tr}(\not{p}'+m)\not{p}\not{k}(-2\not{p}+4m)$

$= 4\cdot4\,[p'\cdot p\; k\cdot p\cdot 2 - p'\cdot k\, m^2] - 2\cdot4\cdot4m^2\, p\cdot k$

$= 32\, p'\cdot p\; k\cdot p - 16\,m^2\, p'\cdot k - 32m^2\, p\cdot k$

line 4 $= \mathrm{tr}(\not{p}'+m)\gamma^\mu\not{k}\gamma^\nu(\not{p}+m)\gamma_\mu\not{k}'\gamma_\nu$

$= \mathrm{tr}(\not{p}+m)\gamma^\mu\not{k}(-2\not{k}'\gamma_\mu\not{p} + 4m\,k'_\mu)$

$= -2\cdot4\,k\cdot k'\,\mathrm{tr}(\not{p}'+m)\not{p} + 4m^2\,\mathrm{tr}\,\not{k}'\not{k}$

$= -8\,k\cdot k'\cdot 4\,p\cdot p' + 4\cdot4m^2\,k\cdot k'$

$= -32\,k\cdot k'\,p\cdot p' + 16m^2\,k\cdot k'$

~ all

$$\begin{aligned}\mathrm{III} = & -16m^4 - 16\,p\cdot p' m^2 - 32\,p'\cdot p\; p\cdot k' + 16m^2 p\cdot k + 32m^2 p\cdot k' \\ & + 32\,p'\cdot p\; k\cdot p - 16m^2\, p\cdot k' - 32m^2\, p\cdot k \\ & - 32\,p\cdot p'\; k\cdot k' + 16m^2\, k\cdot k'\end{aligned}$$

$= 16\,p\cdot p'\{-m^2 - 2p\cdot k' + 2p\cdot k - 2k\cdot k'\}$

$+ 16\{-m^4 - m^2\,p\cdot k + m^2\,p\cdot k' + m^2\,k\cdot k'\}$

$$= 16\, p\cdot p' \left\{ -m^2 - 2\overbrace{(p+k)}^{p'+k'}\cdot k' + 2p\cdot k \right\}$$

$$+ 16 \left\{ -m^4 - m^2 p\cdot k + m^2 p\cdot k' + m^2 (p p' - m^2) \right\}$$

$$= 16\, p\cdot p' \left\{ -m^2 - 2\cancel{p'}k' + 2\cancel{p}\cdot k \right\} + 16 m^2\, p\cdot p'$$

$$- 32 m^4 - 16 m^2\, p\cdot k + 16 m^2\, p\cdot k'$$

$$\mathrm{II} = -32 m^4 - 16 m^2\, p\cdot k + 16 m^2\, p\cdot k'$$

this does agree with $\quad \mathrm{II} = -8\left(4m^4 + m^2(s-m^2) + m^2(u-m^2)\right)$

in all

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4}\left[\frac{32}{4}\right]$$

$$\cdot \left\{ \frac{1}{(p\cdot k)^2}\left(p\cdot k\, p\cdot k' + m^2 p\cdot k + m^4\right) \right.$$

$$+ \frac{1}{(p\cdot k')^2}\left(p\cdot k\, p\cdot k' - m^2 p\cdot k' + m^4\right)$$

$$\left. + \frac{1}{(p\cdot k)(p\cdot k')}\left(-2m^4 - m^2 p\cdot k + m^2 p\cdot k'\right) \right\}$$

$$= 2e^4 \left\{ \frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) \right.$$

$$\left. + m^4\left(\frac{1}{(p\cdot k)} - \frac{1}{(p\cdot k')}\right)^2 \right\}$$

= the CM frame

$$p = (E, 0\ 0\ p) \qquad k = (p\ 0\ 0\ -p)$$

$$p' = (E, p\sin\theta, 0\ p\cos\theta) \qquad k' = (p, -p\sin\theta, 0, -p\cos\theta$$

$$p\cdot k = p(E+p) \qquad E+p = E_{cm}$$

$$p\cdot k' = p(E+p\cos\theta) \qquad p = \frac{s-m_e^2}{2\sqrt{s}}$$

$$\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} = \frac{1}{p}\left(\frac{1}{E+p} - \frac{1}{E+p\cos\theta}\right)$$

$$= -\frac{1}{p}\frac{p\cos\theta}{(E+p)(E+p\cos\theta)}$$

$$= -\frac{\cos\theta}{(E+p)(E+p\cos\theta)}$$

so

$$\frac{1}{4}\sum|\mathcal{M}|^2 = 2e^4\left\{\left(\frac{E+p\cos\theta}{E+p} + \frac{E+p}{E+p\cos\theta}\right) - 2m^2\frac{\cos\theta}{(E+p)(E+p\cos\theta)} + \frac{m^4\cos^2\theta}{(E+p)^2(E+p\cos\theta)^2}\right\}$$

$$d\Pi = \frac{1}{16\pi}\, d\cos\theta \cdot \left(\frac{2p}{E+p}\right)$$

then

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{4Ep\left(\frac{p}{E}+1\right)} \; \frac{1}{16\pi} \; \frac{2p}{(E+p)} \; 2e^4 \left\{ \dots \right\}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{E_{cm}^2} \left\{ \frac{2E^2 + 2Ep(1+\cos\theta) + p^2(1+\cos^2\theta)}{(E+p)(E+p\cos\theta)} \right.$$

$$\left. - \frac{2m^2\cos\theta}{(E+p)(E+p\cos\theta)} + \frac{m^4\cos^2\theta}{(E+p)^2(E+p\cos\theta)^2} \right\}$$

for $E \gg m$

$$\frac{d\sigma}{d\cos\theta} \sim \frac{\pi\alpha^2}{4E^2} \cdot \frac{5 + 2\cos\theta + \cos^2\theta}{2(1+\cos\theta)}$$

2.) a.) In the limit of $m_e \to 0$

$$i\mathcal{M} = -ie^2 \, \bar{u}(p') \Big\{ \gamma\cdot\varepsilon^*(k') \frac{\not{p}+\not{k}}{2p\cdot k} \gamma\cdot\varepsilon(k)$$

$$+ \gamma\cdot\varepsilon(k) \frac{\not{p}-\not{k}'}{-2p\cdot k'} \gamma\cdot\varepsilon^*(k') \Big\} u(p)$$

Note ~~that~~

$$\bar{u}(p') \gamma^\mu \gamma^\nu \gamma^\lambda u(p)$$

$$= u^\dagger(p') \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\lambda \\ \bar{\sigma}^\lambda & 0 \end{pmatrix} u(p)$$

$$= u^\dagger(p') \begin{bmatrix} \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\lambda & 0 \\ 0 & \sigma^\mu \bar{\sigma}^\nu \sigma^\lambda \end{bmatrix} u(p)$$

so left-handed spinors $u_L(p) = \begin{pmatrix} X \\ 0 \end{pmatrix}$ and right-handed spinors $u_R(p) = \begin{pmatrix} 0 \\ X \end{pmatrix}$ do not mix.

so the possible nonzero helicity amplitudes are

$$e^-_L \gamma_L \to e^-_L \gamma_L \qquad e^-_R \gamma_L \to e^-_R \gamma_L$$

$$e^-_L \gamma_L \to e^-_L \gamma_R \qquad e^-_R \gamma_L \to e^-_R \gamma_R$$

$$e^-_L \gamma_R \to e^-_R \gamma_L \qquad e^-_R \gamma_R \to e^-_R \gamma_L$$

$$e^-_L \gamma_R \to e^-_L \gamma_R \qquad e^-_R \gamma_R \to e^-_R \gamma_R$$

first compute $e_L^- \gamma \to e_L^- \gamma$ amplitudes. Then

 $$= -ie^2 \sqrt{2E}\,(-s_2 c_2)\;\bar\sigma\cdot\varepsilon^*(k')\;\frac{\sigma\cdot(p+k)}{2p\cdot k}\;\bar\sigma\cdot\varepsilon(k)\begin{pmatrix}0\\1\end{pmatrix}\sqrt{2E}$$

 $$= +ie^2\sqrt{2E}\,(-s_2 c_2)\;\bar\sigma\cdot\varepsilon(k)\;\frac{\sigma\cdot(p-k')}{2p\cdot k'}\;\bar\sigma\cdot\varepsilon^*(k')\begin{pmatrix}0\\1\end{pmatrix}\sqrt{2E}$$

$s_2 = \sin\theta/2 \quad c_2 = \cos\theta/2$

now $\quad 2p\cdot k = 4E^2 \qquad 2p\cdot k' = 2E^2(1+\cos\theta)$

so $\qquad s = \sin\theta, \quad c = \cos\theta$

 $$= -ie^2\,\frac{1}{2E}\,(-s_2 c_2)\;\bar\sigma\cdot\varepsilon^*(k')\,\sigma\cdot(p+k)\;\bar\sigma\cdot\varepsilon(k)\begin{pmatrix}0\\1\end{pmatrix}$$

 $$= +ie^2\,\frac{1}{E(1+c)}\,(-s_2 c_2)\;\bar\sigma\cdot\varepsilon(k)\,\sigma\cdot(p-k')\;\bar\sigma\cdot\varepsilon^*(k')\begin{pmatrix}0\\1\end{pmatrix}$$

$p+k = (2E, \vec{0}) \quad$ so $\quad \sigma\cdot(p+k) = 2E\cdot\underline{1}$

$p-k' = (E\ 0\ 0\ E) - (E, -Es, 0\ -Ec) = (0, +Es, 0, E(1+c))$

so $\quad \sigma\cdot(p-k') = -E\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}$

so

 $$= -ie^2\,(-s_2 c_2)\;\bar\sigma\cdot\varepsilon^*(k')\;\underline{1}\;\bar\sigma\cdot\varepsilon(k)\begin{pmatrix}0\\1\end{pmatrix}$$

 $$= \frac{-ie^2}{(1+c)}\,(-s_2 c_2)\;\bar\sigma\cdot\varepsilon(k)\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\bar\sigma\cdot\varepsilon^*(k')\begin{pmatrix}0\\1\end{pmatrix}$$

finally we need

$$\bar{\sigma}\cdot\varepsilon(k) = \quad L: \quad \frac{1}{\sqrt{2}}\left(-\sigma^1 - i\sigma^2\right) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0&-1\\-1&0\end{pmatrix} - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right] = -\sqrt{2}\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$R: \quad \frac{1}{\sqrt{2}}\left(-\sigma^1 + i\sigma^2\right) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}0&-1\\-1&0\end{pmatrix} + i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right] = -\sqrt{2}\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$\bar{\sigma}\cdot\varepsilon^*(k') = \quad L: \quad \frac{1}{\sqrt{2}}\left(\left(\begin{pmatrix}0&-c\\-c&0\end{pmatrix} + i\begin{pmatrix}0&-i\\i&0\end{pmatrix} + s\begin{pmatrix}1&\\&-1\end{pmatrix}\right)\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}s&1-c\\-(1+c)&-s\end{pmatrix}$$

$$R: \quad \frac{1}{\sqrt{2}}\left(\begin{pmatrix}0&-c\\-c&0\end{pmatrix} - i\begin{pmatrix}0&-i\\i&0\end{pmatrix} + s\begin{pmatrix}1&\\&-1\end{pmatrix}\right)\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}s&-(1+c)\\(1-c)&-s\end{pmatrix}$$

again:

| $\bar{\sigma}\cdot\varepsilon_L(k)$ | $\bar{\sigma}\cdot\varepsilon_R(k)$ | $\bar{\sigma}\cdot\varepsilon^*_L(k')$ | $\bar{\sigma}\cdot\varepsilon^*_R(k')$ |
|---|---|---|---|
| $-\sqrt{2}\begin{pmatrix}0&1\\0&0\end{pmatrix}$ | $-\sqrt{2}\begin{pmatrix}0&0\\1&0\end{pmatrix}$ | $\frac{1}{\sqrt{2}}\begin{pmatrix}s&1-c\\-(1+c)&-s\end{pmatrix}$ | $\frac{1}{\sqrt{2}}\begin{pmatrix}s&-(1+c)\\1-c&-s\end{pmatrix}$ |

Begin with $\mathcal{H}$. note that $\bar{\sigma}\cdot\varepsilon_R(k)\begin{pmatrix}0\\1\end{pmatrix} = 0$ so

$$\mathcal{M}(\bar{e}_L\gamma_R \to \bar{e}_L\gamma_{L\,\text{or}\,R}) = 0$$

$$i\mathcal{M}(\bar{e}_L\gamma_L \to \bar{e}_L\gamma_L) = -ie^2\,(-s_2 c_2)\,\frac{1}{\sqrt{2}}\begin{pmatrix}s&(1-c)\\-(1+c)&-s\end{pmatrix}(-\sqrt{2})\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= ie^2\,(-s_2c_2)\begin{pmatrix}s&(1-c)\\-(1+c)&-s\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= ie^2\left[-s_2 s - c_2(1+c)\right]$$

$$= +ie^2\left[-\cos\theta/2 - (\cos\theta\cos\theta/2 + \sin\theta\sin\theta/2)\right]$$

$$= -2ie^2\cos\theta/2$$

$$i\mathcal{M}(e^-_L \gamma_L \to e^-_L \gamma_R) = -ie^2 (-s_2 c_2) \tfrac{1}{\sqrt{2}} \begin{pmatrix} s & -(1+c) \\ 1-c & -s \end{pmatrix} (-\sqrt{2}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= ie^2 (-s_2 c_2) \begin{pmatrix} s & -(1+c) \\ 1-c & -s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= ie^2 (-s_2 s + c_2 - c_2 c)$$

$$= ie^2 (c_2 - (c_2 c + s_2 s)) = 0$$

so for

$$i\mathcal{M}(e^-_L \gamma_L \to e^-_L \gamma_L) = -2ie^2 c_2$$

all other helicity amplitudes are 0.

for  :

$e^-_L \gamma_L \to e^-_L \gamma_L$ :

$$= \frac{-ie^2}{1+c} (-s_2 c_2)(-\sqrt{2}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} s & 1-c \\ -(1+c) & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{+ie^2}{1+c} (0, -s_2) \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \begin{pmatrix} 1-c \\ -s \end{pmatrix}$$

$$= \frac{+ie^2}{1+c} \left[ -s_2 s (1-c) - s_2 s (1+c) \right]$$

$$= \frac{-2ie^2}{1+c} s_2 s$$

$e_L^- \gamma_L \to e_L^- \gamma_R$ : $\quad \frac{-ie^2}{1+c} (-s_2 c_2) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (-\sqrt{2}) \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} s & -(1+c) \\ (1-c) & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \frac{ie^2}{1+c} (0 \ -s_2) \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \begin{pmatrix} -(1+c) \\ -s \end{pmatrix}$$

$$= \frac{ie^2}{1+c} \left[ + s_2 s (1+c) - s_2 s (1+c) \right] = 0$$

$e_L^- \gamma_R \to e_L^- \gamma_L$ $\quad \frac{-ie^2}{1+c} (-s_2 c_2) -\sqrt{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} s & 1-c \\ -(1+c) & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \frac{+ie^2}{1+c} (c_2, 0) \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \begin{pmatrix} 1-c \\ -s \end{pmatrix}$$

$$= \frac{+ie^2}{1+c} \left[ c_2 \underbrace{(1+c)(1-c)}_{\sin^2\theta} - c_2 s^2 \right] = 0$$

$e_L^- \gamma_R \to e_L^- \gamma_R$ $\quad \frac{-ie^2}{1+c} (-s_2 c_2)(-\sqrt{2}) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} s & -(1+c) \\ (1-c) & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \frac{+ie^2}{1+c} (+c_2, 0) \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \begin{pmatrix} -(1+c) \\ -s \end{pmatrix}$$

$$= \frac{ie^2}{1+c} \left[ -c_2 (1+c)^2 - c_2 s^2 \right]$$

$$= \frac{ie^2}{\cancel{1+c}} \left[ -c_2 (1+c) - c_2 (1-c) \right] \cancel{(1+c)}$$

$$= -2ie^2 c_2$$

now add the contributions

 +  =

$e^-_L \gamma_L \to e^-_L \gamma_L \qquad -2ie^2 \left[ c_2 + \frac{s s_2}{(1+c)} \right]$

$$= -2ie^2 \; \frac{c_2 + (c_2 c + s_2 s)}{(1+c)}$$

$$= -4ie^2 \; \frac{c_2}{1+c}$$

$c c_2 + s s_2 = \cos \theta/2 = c_2$

$c_2^2 = \frac{1+c}{2}$

$$= -2ie^2 \frac{1}{c_2}$$

$e^-_L \gamma_L \to e^-_L \gamma_R \qquad = 0$

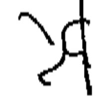
$e^-_L \gamma_R \to e^-_R \gamma_L \qquad = 0$

$e^-_L \gamma_R \to e^-_L \gamma_R \qquad = 0 - 2ie^2 c_2$

$$= -2ie^2 c_2$$

The amplitudes for $e^-_R \gamma \to e^-_R \gamma$ should follow the same pattern. The basic ingredients are:

 $= -ie^2 \sqrt{2E} \, (c_2 \; s_2) \, \sigma \cdot \varepsilon^*(k') \, \frac{\bar\sigma \cdot p+k}{2p\cdot k} \, \sigma \cdot \varepsilon(k) \begin{pmatrix}1\\0\end{pmatrix} (\sqrt{2E})^*$

 $= +ie^2 \sqrt{2E} \, (c_2 \; s_2) \, \sigma \cdot \varepsilon(k) \, \frac{\bar\sigma \cdot p-k'}{2p\cdot k'} \, \sigma \cdot \varepsilon^*(k') \begin{pmatrix}1\\0\end{pmatrix} \sqrt{2E}$

now $\quad \bar\sigma\cdot(p+k) = 2E \cdot \underline{1} \qquad \bar\sigma\cdot(p-k') = +E\begin{pmatrix} 1+c & s \\ s & -(1+c)\end{pmatrix}$

so

 $= -ie^2 \,(c_2\; s_2)\; \sigma\cdot\varepsilon^*(k')\; 1\; \sigma\cdot\varepsilon(k)\; \begin{pmatrix}1\\0\end{pmatrix}$

 $= \dfrac{+ie^2}{1+c}\,(c_2\; s_2)\; \sigma\cdot\varepsilon(k)\begin{pmatrix} 1+c & s \\ s & -(1+c)\end{pmatrix}\sigma\cdot\varepsilon^*(k')\begin{pmatrix}1\\0\end{pmatrix}$

Evaluate these using:

| $\sigma\cdot\varepsilon_L(k)$ | $\sigma\cdot\varepsilon_R(k)$ | $\sigma\cdot\varepsilon_L^*(k')$ | $\sigma\cdot\varepsilon_R^*(k')$ |
|---|---|---|---|
| $+\sqrt2\begin{pmatrix}0&1\\0&0\end{pmatrix}$ | $+\sqrt2\begin{pmatrix}0&0\\1&0\end{pmatrix}$ | $-\frac{1}{\sqrt2}\begin{pmatrix}s & 1-c\\ -(1+c) & -s\end{pmatrix}$ | $-\frac{1}{\sqrt2}\begin{pmatrix}s & -(1+c)\\ (1-c) & -s\end{pmatrix}$ |

for  $\sigma\cdot\varepsilon_L(k)\begin{pmatrix}1\\0\end{pmatrix} = 0$, so the possibly nonzero amplitudes are:

$e^-_R\gamma_R \to e^-_R\gamma_L$ : $\quad -ie^2\,(c_2 s_2)\left(-\frac{1}{\sqrt2}\right)\begin{pmatrix}s & 1-c\\ -(1+c) & -s\end{pmatrix}\sqrt2\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$$= ie^2\,(c_2 s_2)\begin{pmatrix}s & 1-c\\ -(1+c) & -s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= ie^2\left[c_2(1-c) - s_2 s\right] = ie^2\left[c_2 - (cc_2 + ss_2)\right]$$
$$= 0$$

$e^-_R\gamma_R \to e^-_R\gamma_R$ : $\quad -ie^2\,(c_2 s_2)\left(-\frac{1}{\sqrt2}\right)\begin{pmatrix}s & -(1+c)\\ 1-c & -s\end{pmatrix}\sqrt2\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$$= ie^2(c_2 s_2)\begin{pmatrix}s & -(1+c)\\ (1-c) & -s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= ie^2\left[-c_2(1+c) - ss_2\right] = -2ie^2\, c_2$$

for $\chi$

$e^-_R \gamma_L \to e^-_R \gamma_L$: $\quad \dfrac{+ie^2}{1+c} \, (c_2 \; s_2) \, \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \left(-\dfrac{1}{\sqrt{2}}\right) \begin{pmatrix} s & (1+c) \\ -(1+c) & -s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \frac{-ie^2}{1+c} \, (0, c_2) \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \begin{pmatrix} s \\ -(1+c) \end{pmatrix}$$

$$= \frac{ie^2}{1+c} \left[ c_2 s^2 + c_2 (1+c)^2 \right]$$

$$= -\frac{ie^2}{1+c} (1+c) \left[ c_2(1-c) + c_2(1+c) \right] = 2ie^2 c_2$$

$e^-_R \gamma_L \to e^-_R \gamma_R$: $\quad \dfrac{+ie^2}{1+c} \, (c_2 s_2) \, \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \left(\dfrac{-1}{\sqrt{2}}\right) \begin{pmatrix} s & -(1+c) \\ (1-c) & -s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \frac{-ie^2}{1+c} \, (c_2 s_2) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \begin{pmatrix} s \\ (1-c) \end{pmatrix}$$

$$= \frac{-ie^2}{1+c} \left[ c_2 s^2 - c_2 \underbrace{(1+c)(1-c)}_{s^2} \right] = 0$$

$e^-_R \gamma_R \to e^-_R \gamma_L$ $\quad \dfrac{+ie^2}{1+c} \, (c_2 s_2) \, \sqrt{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \left(\dfrac{-1}{\sqrt{2}}\right) \begin{pmatrix} s & (1-c) \\ -(1+c) & -s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \frac{-ie^2}{1+c} \, (s_2, 0) \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \begin{pmatrix} s \\ -(1+c) \end{pmatrix}$$

$$= \frac{-ie^2}{1+c} \left[ s_2 s (1+c) - s_2 s (1+c) \right] = 0$$

$e^-_R \gamma_R \to e^-_R \gamma_R$ : $\quad \frac{+ie^2}{1+c}(c_2\ s_2)\sqrt{2}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\left(\frac{-1}{\sqrt{2}}\right)\begin{pmatrix}s & -(1+c)\\ (1-c) & -s\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$$= \frac{-ie^2}{1+c}(s_2, 0)\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\begin{pmatrix}s\\ (1-c)\end{pmatrix}$$

$$= \frac{-ie^2}{1+c}\left[s_2(1+c)s + s_2 s(1-c)\right]$$

$$= \frac{-2ie^2}{1+c}\, s s_2$$

the complete amplitude for $e^-_R\gamma_R \to e^-_R\gamma_R$ is:

$$-2ie^2 c_2 - \frac{2ie^2}{1+c} s s_2 = \frac{-2ie^2}{1+c}\left[c_2 + c c_2 + s s_2\right]$$

$$= -2ie^2 \frac{2c_2}{1+c} = -2ie^2 \frac{1}{c_2}$$

so!

$$i\mathcal{M}(e^-_L\gamma_L \to e^-_L\gamma_L) = i\mathcal{M}(e^-_R\gamma_R \to e^-_R\gamma_R) = -2ie^2\frac{1}{c_2}$$

$$i\mathcal{M}(e^-_L\gamma_R \to e^-_L\gamma_R) = i\mathcal{M}(e^-_R\gamma_L \to e^-_R\gamma_L) = -2ie^2 c_2$$

all other helicity amplitudes vanish.

then the polarization summed/averaged cross section is:

$$\frac{d\sigma}{d\cos\Theta} = \frac{1}{2s} \cdot \frac{1}{16\pi} \cdot \frac{1}{4} \sum |M|^2$$

$$= \frac{1}{2s} \; \frac{1}{16\pi} \; \frac{1}{4} \cdot 4e^4 \cdot 2 \cdot \left[ \frac{1}{c_2^2} + c_2^2 \right]$$

$$= \frac{\pi\alpha^2}{E_{cm}^2} \left[ \frac{2}{1+\cos\Theta} + \frac{1+\cos\Theta}{2} \right]$$

$$= \frac{\pi\alpha^2}{4E^2} \; \frac{4 + (1+\cos\Theta)^2}{2(1+\cos\Theta)}$$

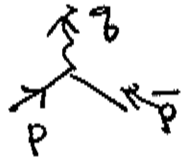
which reproduces the result on p. 9.

3.) The vertex for this problem is

$$= -ig\,\gamma^\mu$$

I will ignore the electron mass except where it is essential in part (c).

a.) $\sigma(e^+e^- \to B)$ is given by



$$= -ig\, \bar{v}(\bar{p})\, \gamma^\mu u(p) \cdot \varepsilon_\mu^*(q)$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{g^2}{4}\,\mathrm{tr}\left[\bar{\not p}\,\gamma^\mu \not p\,\gamma^\nu\right](-g_{\mu\nu})$$

$$= -\frac{g^2}{4}\,\mathrm{tr}\,\bar{\not p}\,(-2\not p)$$

$$= \frac{g^2}{4}\cdot 2\cdot 4\;\bar p\cdot p = g^2\, 2\bar p\cdot p = g^2(\bar p + p)^2$$

$$= g^2 q^2 = M^2\cdot g^2$$

$$\sigma(e^+e^-\to B) = \frac{1}{2s}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2E_q}(2\pi)^4\delta(p+\bar p - q)\left(\frac{1}{4}\sum|\mathcal{M}|^2\right)$$

$$= \frac{1}{2s}\;2\pi\;\frac{1}{2M}\;\delta(p^0+\bar p^0 - M)\cdot g^2 M^2$$

CM frame.

$$= \frac{1}{2s}\cdot\;2\pi\;\delta\left((p+\bar p)^2 - M^2\right)\,g^2M^2$$

$s = M^2$

$$= \pi g^2\;\delta(s - M^2)$$

$$\Gamma(B\to e^+e^-) = \frac{1}{2M}\int d\Pi_2\;\left(\frac{1}{3}\sum|\mathcal{M}|^2\right)$$

↑ spin average for spin-1

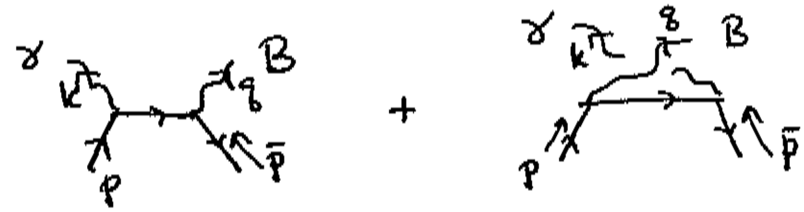
$\sum|\mathcal{M}|^2$ is the same as above $= 4g^2M^2$

$$= \frac{1}{2M}\cdot\frac{1}{8\pi}\cdot\frac{4}{3}\cdot g^2M^2 = \frac{g^2 M}{12\pi}$$

so

$$\left.\begin{aligned}\sigma(e^+e^-\to B) &= \pi g^2\,\delta(s-M^2)\\ \Gamma_B &= \frac{g^2M}{12\pi}\end{aligned}\right\}\quad \sigma(e^+e^-\to B) = \frac{12\pi^2}{M}\Gamma_B\,\delta(s-M^2)$$

b.) for $e^+e^- \to \gamma + B$:




$$= (-ig)(-ie)\bar{v}(\bar{p})\Big\{ \gamma\cdot\varepsilon^*(q)\, \frac{i(\not p - \not k)}{(p-k)^2}\, \gamma\cdot\varepsilon^*(k)$$

$$+ \gamma\cdot\varepsilon^*(k)\, \frac{i\,\not k - \not{\bar p}}{(\bar p - k)^2}\, \gamma\cdot\varepsilon^*(q) \Big\}\, u(p)$$

$$(p-k)^2 = p^2 - 2p\cdot k + k^2 = -2p\cdot k$$
$$(\bar p - k)^2 = -2\bar p\cdot k$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{e^2g^2}{4}\,\mathrm{tr}\Big\{ \not{\bar p}\Big[\frac{\gamma^\mu(\not p - \not k)\gamma^\nu}{2p\cdot k} + \frac{\gamma^\nu(\not k - \not{\bar p})\gamma^\mu}{2\bar p\cdot k}\Big]$$

$$\not p\Big[\frac{\gamma_\nu(\not p - \not k)\gamma_\mu}{2p\cdot k} + \frac{\gamma_\mu(\not k - \not{\bar p})\gamma_\nu}{2\bar p\cdot k}\Big]\Big\}$$

$$= \frac{e^2g^2}{4}\Big\{ \frac{\mathrm{I}}{(2p\cdot k)^2} + \frac{\mathrm{II}}{2p\cdot k\; 2\bar p\cdot k} + \frac{\mathrm{III}}{2\bar p\cdot k\; 2p\cdot k} + \frac{\mathrm{IV}}{(2\bar p\cdot k)^2}\Big\}$$

$$p^2 = \bar{p}^2 = k^2 = 0 \qquad q^2 = M^2$$

$$\begin{aligned}
\mathrm{I} &= \mathrm{tr}\ \bar{\not p}\, \gamma^\mu (\not p - \not k) \gamma^\nu \not p\, \gamma_\nu (\not p - \not k) \gamma_\mu \\
&= \mathrm{tr}\ \gamma_\mu \bar{\not p}\, \gamma^\mu (\not p - \not k) \gamma^\nu \not p\, \gamma_\nu (\not p - \not k) \\
&= (-2)^2\ \mathrm{tr}\ \bar{\not p} (\not p - \not k) \not p (\not p - \not k) \\
&= 4 \cdot 4 \cdot [\, \bar{p}\cdot(p-k)\ p\cdot(p-k) \cdot 2 \ - \ p\cdot\bar{p}\ (p-k)^2 \,] \\
&= 16 \cdot [\, -2\bar{p}\cdot p\ p\cdot k + 2\bar{p}\cdot k\ pk + 2 p\cdot\bar{p}\ p\cdot k \,] \\
&= 32\, [p\cdot k]\, [\, \cancel{-\bar{p}\cdot p} + \bar{p}\cdot k + \cancel{\bar{p}\cdot p} \,] \\
&= 32\ p\cdot k\ \bar{p}\cdot k
\end{aligned}$$

$$\begin{aligned}
\mathrm{IV} &= \mathrm{tr}\ \bar{\not p}\, \gamma^\nu (\bar{\not p} - \not k) \gamma^\mu \not p\, \gamma_\mu (\bar{\not p} - \not k) \gamma_\nu \\
&= (-2)^2\ \mathrm{tr}\ \bar{\not p} (\bar{\not p} - \not k) \not p (\bar{\not p} - \not k) \\
&= 4 \cdot 4 \cdot \{\, \bar{p}\cdot(\bar{p}-k)\ p\cdot(\bar{p}-k) \cdot 2 \ - \ p\cdot\bar{p}\ (\bar{p}-k)^2 \,\} \\
&= 16\ \{\, -\bar{p}\cdot k\, (\cancel{2p\bar{p}} - 2p\cdot k) \ - \ (\cancel{p\cdot\bar{p}})(\cancel{-2\bar{p}\cdot k}) \,\} \\
&= 32\ \bar{p}\cdot k\ p\cdot k
\end{aligned}$$

$$\begin{aligned}
\mathrm{II} &= \mathrm{tr}\ \bar{\not p}\, \gamma^\mu (\not p - \not k) \gamma^\nu \not p\, \gamma_\mu (\bar{\not p} - \not k) \cdot (-1)\, \gamma_\nu \\
&= (-1)\ \mathrm{tr}\ \bar{\not p}\, \gamma^\mu (\not p - \not k) (-2) (\bar{\not p} - \not k) \gamma_\mu \not p \\
&= 2\ \mathrm{tr}\ \bar{\not p}\ 4 (p-k)(\bar{p}-k) \not p \\
&= 2 \cdot 4 \cdot 4\, (p\cdot\bar{p})\, (\, p\cdot\bar{p} - p\cdot k - \bar{p}\cdot k \,)
\end{aligned}$$

now $\quad s = 2p\cdot\bar{p} \qquad t = -2p\cdot k \qquad u = -2\bar{p}\cdot k$

$$s + t + u = M^2 \quad \text{so} \quad p\cdot\bar{p} - p\cdot k - \bar{p}\cdot k = M^2/2$$

$$\mathrm{II} = 16\, p\cdot\bar{p}\;\; M^2$$

$$\begin{aligned}
\mathrm{III} &= \mathrm{tr}\; \bar{\not{p}}\; \gamma^\nu (\bar{\not{p}} - \not{k}) (-1) \gamma^\mu \not{p}\; \gamma_\nu (\not{p} - \not{k}) \gamma_\mu \\
&= \mathrm{tr}\; (-1)(-2)\; \bar{\not{p}}\; \gamma^\nu (\bar{p} - k)(p - k) \gamma_\nu \not{p} \\
&= 2\cdot 4\;\; \mathrm{tr}\; \bar{\not{p}}\;\; (\bar{p} - k)(p - k)\; \not{p} \\
&= 2\cdot 4\cdot 4\;\; \bar{p}\cdot p\;\; (\bar{p} - k)(p - k) \;=\; 16\, p\cdot\bar{p}\; M^2
\end{aligned}$$

in all:

$$\frac{1}{4}\sum |\mathcal{M}|^2 = \frac{e^2 g^2}{4\cdot 4} \left\{ \frac{32\, p\cdot k\; \bar{p}\cdot k}{(p\cdot k)^2} + \frac{32\, p\cdot k\; \bar{p}\cdot k}{(\bar{p}\cdot k)^2} \right.$$

$$\left. + \; 2\; \frac{16\, p\cdot\bar{p}\; M^2}{(p\cdot k)(\bar{p}\cdot k)} \right\}$$

$$= 2 g^2 e^2 \left\{ \frac{\bar{p}\cdot k}{p\cdot k} + \frac{p\cdot k}{\bar{p}\cdot k} + \frac{p\cdot\bar{p}\; M^2}{p\cdot k\;\; \bar{p}\cdot k} \right\}$$

In the CM system:

$$p = (E, 0, 0, E) \qquad\qquad k = (p, p\sin\theta, 0, p\cos\theta)$$

$$\bar{p} = (E, 0\; 0\; -E) \qquad\qquad q = (\mathcal{E}, -p\sin\theta, 0, -p\cos\theta)$$

$$\text{s.t.} \qquad \mathcal{E}^2 - p^2 = M^2 \qquad\qquad \mathcal{E} + p = 2E$$

$$p = \frac{4E^2 - M^2}{2E} \qquad\qquad \mathcal{E} = \frac{4E^2 + M^2}{2E}$$

$$p\cdot k = Ep(1-\cos\theta) \qquad \bar{p}\cdot k = Ep(1+\cos\theta)$$

$$p\cdot\bar{p} = 2E^2$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = 2e^2g^2\left\{\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta} + \frac{2E^2M^2}{E^2p^2(1-\cos^2\theta)}\right\}$$

$$= 4e^2g^2\left\{\frac{1+\cos^2\theta}{1-\cos^2\theta} + \frac{(M^2/p^2)}{(1-\cos^2\theta)}\right\}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{8E^2}\,\frac{1}{16\pi}\,\frac{2p}{2E}\,4e^2g^2\left\{\frac{M^2/p^2 + (1+\cos^2\theta)}{1-\cos^2\theta}\right\}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2E^3p}\,\pi\alpha\cdot\frac{g^2}{4\pi}\cdot\left(\frac{M^2 + p^2(1+\cos^2\theta)}{1-\cos^2\theta}\right)$$

where, again $\quad p = \dfrac{4E^2-M^2}{2E}$

c.) Restoring the electron mass in the t-channel propagator, the denominator is

$$(p-k)^2 - m_e^2 = p^2 - 2p\cdot k - m_e^2 = -2p\cdot k$$

but not $\quad p = (E, 0, 0, p_e) \qquad p_e = (E^2-m_e^2)^{1/2} \cong E - \dfrac{m_e^2}{2E} + \dots$

$$p\cdot k = (E - p_e\cos\theta)\,p$$

Using $\quad p_e \cong E - \dfrac{m_e^2}{2E} + \cdots$

$$p\cdot k = p E\left\{(1-\cos\Theta) + \frac{m_e^2}{2E^2}\cos\Theta + \cdots\right\}$$

$$\cong p E \left(1 - \cos\Theta + \frac{m_e^2}{2E^2}\right) \quad \text{as} \quad \cos\Theta \to 1$$

then

$$\int_0 d\cos\Theta \, \frac{1}{1-\cos\Theta} \to \int_0 d\cos\Theta \, \frac{1}{\left(1-\cos\Theta + \frac{m_e^2}{2E^2}\right)}$$

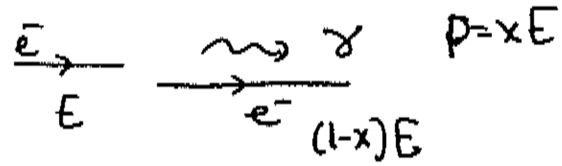
$$= \log \frac{2E^2}{m_e^2}$$

~~integrate~~ $\dfrac{d\sigma}{d\cos\Theta}$ over forward $\gamma$'s, we find

$\sigma(e^+e^- \to \text{forward } \gamma + B)$

$$\cong \frac{\pi\alpha}{2E^3 p} \frac{g^2}{4\pi} \frac{M^2 + p^2 \cdot 2}{2} \log\left(\frac{2E^2}{m_e^2}\right)$$

let $\quad p = xE$




$x$ is fixed by the condition that

$\underset{(1-x)E}{\xrightarrow{e^-}} \quad \underset{E}{\xleftarrow{e^+}}$ should add to a mass $M$

$$M^2 = ((1-x)p + \bar{p})^2 = 2(1-x)\, p\cdot\bar{p}$$

or $$(1-x) = M^2/s$$

$$p^2 = x^2 E^2 = \frac{x^2 s}{4}$$

and replace $\lg \frac{2E^2}{m_e^2} \to \lg \frac{s}{m_e^2}$ with small error.

then

$\sigma(e^+e^- \longrightarrow \text{forward } \gamma + B)$

$$= \frac{\pi\alpha}{2E^4 x} \frac{g^2}{4\pi} \frac{1}{2} \left[(1-x) + x^2/2\right]\cdot s \cdot \lg \frac{s}{m_e^2}$$

$$= \int_0^1 dx \cdot s \cdot \pi g^2\, \delta(M^2 - (1-x)s)$$

$$\cdot \quad \frac{\alpha}{s^2 x} \frac{1}{2\pi} (2 - 2x + x^2)\cdot s \cdot \lg \frac{s}{m_e^2}$$

$$= \int_0^1 dx \quad \pi g^2\, \delta(M^2 - (1-x)s)$$

$$\cdot \quad \frac{\alpha}{2\pi} \frac{1}{x} \left[1 + (1-x)^2\right] \lg \frac{s}{m_e^2}$$

$$= \int_0^1 dx \left\{ \frac{\alpha}{2\pi} \frac{1}{x} \left[1 + (1-x)^2\right] \lg \frac{s}{m_e^2} \right\} \left[\sigma(e^+e^- \to B)\right]_{E_{cm}^2 = (1-x)s}$$