Physics 330 – Problem Set #5

Solutions

1.) a) Let the source be turned on at a time t_{on} and turned off at t_{off}. Let t_1, t_2 be any times s.t.

$$t_1 > t_{off} > t_{on} > t_2$$

Then for $t_1 \geq t_{off}$ and for $t_2 \leq t_{on}$, the ground state of H is the free-particle vacuum $|0\rangle$.

We can compute $e^{-iH(t_1-t_2)}$ by

$$U(t_1,t_2) = e^{iH_0 t_1} e^{-iH(t_1-t_2)} e^{-iH_0 t_2}$$

$$= T\left\{ e^{-i\int_{t_2}^{t_1} dt\, H_I(t)} \right\}$$

$$= T\left\{ e^{+i\int d^4x\, j(x)\, \phi_I(x)} \right\}$$

Now we can ignore the limits in time, since $j(x)$ is nonzero only for $t_1 < t < t_2$

The probability that no particles are produced while j is turned on is

$$P(0) = \left| \langle 0| e^{-iH(t_1-t_2)} |0\rangle \right|^2 = \left| \langle 0| T\{ e^{i\int d^4x\, j\,\phi_I} \} |0\rangle \right|^2$$

now

$$\langle 0|\, T\{e^{i\int d^4x\; j(x)\,\phi_I(x)}\}\,|0\rangle$$

$$= \langle 0|\, T\{ 1 + i\int d^4x\; j(x)\,\phi_I(x) - \frac{1}{2}\int d^4x\, d^4y\; j(x)\,j(y)\,\phi_I(x)\,\phi_I(y) + \cdots \}\,|0\rangle$$

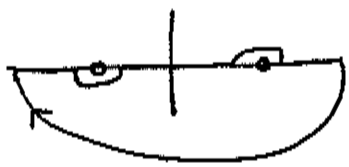
$\langle 0|\phi_I(x)|0\rangle = 0$ $\qquad$ $\langle 0|\, T\{\phi_I(x)\,\phi_I(y)\}\,|0\rangle = \overline{\phi_I(x)\,\phi_I(y)}$

$$= \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\varepsilon}\, e^{-ip\cdot(x-y)}$$

$$= 1 - \frac{1}{2}\int d^4x\, d^4y\; j(x)\,j(y)\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\varepsilon}\, e^{-ip(x-y)} + \cdots$$

$$= 1 - \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\varepsilon}\,\tilde{j}(-p)\,\tilde{j}(p) + \cdots$$

since $j(x)$ has compact support, $\tilde{j}(p)$ is an entire function and we can evaluate the dp^0 integral by contours:



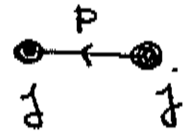
$$= \frac{-2\pi i\; i}{2\pi\cdot 2E_p}\Big|_{p^0=E_p}$$

$$= 1 - \frac{1}{2}\underbrace{\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\,|\tilde{j}(p)|^2\Big|_{p^0=E_p}}_{\text{call this } = \lambda} + \cdots$$

$$= 1 - \frac{1}{2}\lambda + \cdots$$

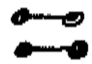
then

$$P(0) = \left|1 - \tfrac{1}{2}\lambda + \cdots\right|^2 = 1 - \lambda + \cdots$$

c.) Represent $\int d^4x\, j(x)\, \phi_J(x) \int d^4y\, j(y)\, \phi_J(y) =$ 

Then

$$\langle 0 | T\{ e^{i\int d^4x\, j(x)\phi_J(x)} \} | 0 \rangle$$

$= 1 + \frac{(-i)^2}{2!}$  $+ \frac{(-i)^4}{4!} \cdot C_4$  $+ \frac{(-i)^6}{6!} C_6$ 

$+ \cdots$

$$= 1 - \tfrac{1}{2}\lambda + \tfrac{1}{4!} C_4 \lambda^2 - \tfrac{1}{6!} C_6 \lambda^3 + \cdots$$

$C_4, C_6 ..$ are the numbers of possible contractions

$$C_4 \qquad \phi_J\phi_J\phi_J\phi_J \qquad \phi_J\phi_J\phi_J\phi_J \qquad \phi_J\phi_J\phi_J\phi_J$$

$$= 3 \cdot 1 = 3 \qquad\qquad \frac{C_4}{4!} = \frac{1}{4 \cdot 2} = \frac{1}{2^2} \cdot \frac{1}{2}$$

$$C_6 = 5 \cdot 3 \cdot 1 \qquad \frac{C_6}{6!} = \frac{1}{6 \cdot 4 \cdot 2} = \frac{1}{2^3} \cdot \frac{1}{3!}$$

$$C_8 = 7 \cdot 5 \cdot 3 \cdot 1 \qquad \frac{C_8}{8!} = \frac{1}{8 \cdot 6 \cdot 4 \cdot 2} = \frac{1}{2^4} \frac{1}{4!}$$

$$\text{so} \quad = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{2^n n!} = \exp\left[-\frac{\lambda}{2}\right]$$

then $\quad P(0) = e^{-\lambda}$

d.) The probability of producing 1 particle is

$$\int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\left|\langle k|\,T\{e^{i\int j\phi_I}\}\,|0\rangle\right|^2$$

$$\langle k|\,T\{e^{i\int j\phi_I}\}\,|0\rangle \cong \langle k|\; i\int d^4x\, j(x)\,\phi_I(x)\;|0\rangle + \cdots$$

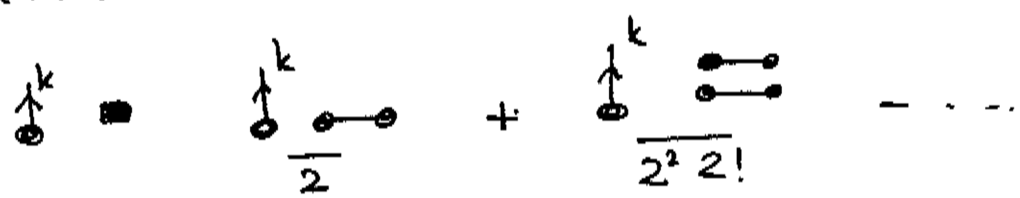
$$= i\int d^4x\; e^{ik\cdot x}\, j(x) + \cdots = i\,\tilde{j}(k) + \cdots$$

so

$$P_1 = \int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\,|\tilde{j}(k)|^2 + \cdots = \lambda + \cdots$$

represent the matrix element by a Feynman diagram:

The whole series is:



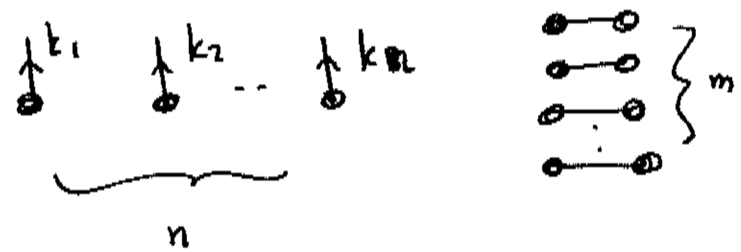
$$= i\tilde{j}(k)\cdot\left[1 - \frac{\lambda}{2} + \frac{\lambda^2}{2^2\,2!} + \cdots\right]$$

then

$$\mathcal{P}_1 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left(\tilde{j}(k)\, e^{-\lambda/2} \right)^2$$

$$= \lambda\, e^{-\lambda}$$

e.) A typical diagram for producing n particles is



$$\cdot \frac{1}{2^m\, m!}$$

square this and integrate over phase space

$$\frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{2E_1} \quad \text{—} \quad \int \frac{d^3k_n}{(2\pi)^3} \frac{1}{2E_n}$$

The $\frac{1}{n!}$ must be there because these are identical Bose-Einstein particles. We sum over inequivalent configurations only. Then

$$\mathcal{P}_n = \frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{2E_1} \cdots \int \frac{d^3k_n}{(2\pi)^3} \frac{1}{2E_n} \cdot |\tilde{j}(k_1)|^2 \cdots |\tilde{j}(k_n)|^2 \left(e^{-\lambda/2}\right)^2$$

$$= \frac{\lambda^n}{n!}\, e^{-\lambda}$$

f:) $$\sum_n P_n = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} = e^{\lambda} e^{-\lambda} = 1$$

$$\langle N \rangle = \sum_n n P_n = \sum_{n=\text{[redacted]}}^{\infty} \frac{\lambda^n}{(n-1)!} e^{-\lambda} = \lambda \, e^{\lambda} e^{-\lambda} = \lambda$$

$$\langle N^2 \rangle = \sum_n n^2 P_n = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \lambda^n e^{-\lambda} = \sum_{n=1}^{\infty} \frac{(n-1)+1}{(n-1)!} \lambda^n e^{-\lambda}$$

$m = (n-1)$ $$= \left(\sum_{m=0}^{\infty} \frac{1}{m!} \lambda^{m+1} + \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \lambda^{m+1} \right) e^{-\lambda}$$

$$= (\lambda e^{\lambda} + \lambda^2 e^{\lambda}) e^{-\lambda}$$

so

$$\langle N^2 \rangle = \lambda(\lambda+1) = \lambda^2 + \lambda$$

$$\langle (N - \langle N \rangle)^2 \rangle = \langle N^2 - 2N\langle N \rangle + \langle N \rangle^2 \rangle$$

$$= \langle N^2 \rangle - (\langle N \rangle)^2$$

$$= \lambda$$

2.) a) $$\mathcal{H} = \int d^3x \left\{ \tfrac{1}{2}\pi^{i\,2} + \tfrac{1}{2}(\vec{\nabla}\Phi^i)^2 + \tfrac{1}{2}m^2\Phi^{i2} + \frac{\lambda}{4}((\Phi^i)^2)^2 \right\}$$

$$= \sum_{i=1}^{N} H_0^i + \frac{\lambda}{4}\Big(\sum_{i=1}^{N}(\Phi^i)^2\Big)^2$$

H_0^i is a Klein-Gordon Hamiltonian so

$$\langle 0| T\{\Phi_I^i(x)\,\Phi_I^j(y)\}|0\rangle = \delta^{ij} \times \text{Klein-Gordon propagator}$$

$$= \delta^{ij} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

the vertex comes from contracts

$$\Big\langle \; \Big| -i\frac{\lambda}{4}\int d^4x\; \Phi^i\Phi^i\,\Phi^j\Phi^j \;\Big| \Big\rangle \qquad i,j = 1 - N$$

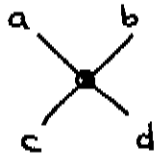
a given contraction is of the form

$$\langle a\, b| -i\frac{\lambda}{4}\int d^4x\; \Phi^i\Phi^i\,\Phi^j\Phi^j\;|c\; d\rangle$$

$$= -i\frac{\lambda}{4}\,\delta^{ai}\delta^{bi}\delta^{cj}\delta^{dj} = -i\frac{\lambda}{4}\,\delta^{ab}\delta^{cd}$$

three different structures are possible. There are $4!$ contractions in all, so each structure has $4!/3 = 8$ possible contractions.

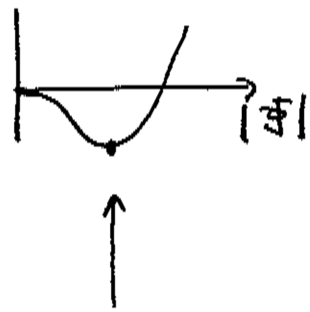
so



$$= -2i\lambda\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right)$$

b.) Now let $m^2 = -\mu^2 < 0$

$$V(\Phi^2) = -\frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4}(\Phi^2)^2$$



$$\frac{\partial V}{\partial \Phi^i} = -\mu^2\Phi^i + \lambda\,\Phi^2\Phi^i$$

so the minimum occurs at $\Phi^2 = v^2 = \frac{\mu^2}{\lambda}$

Choose the orientation $\Phi = (0 \cdots 0, v)$

so $\Phi^i = \pi^i \qquad i = 1 - (N-1) \qquad \langle\pi^i\rangle = 0$

$\Phi^N = v + \sigma \qquad \langle\Phi^N\rangle = v$

expanding about this point: $\Phi^2 = \pi^2 + (v+\sigma)^2$

$$V(\Phi^2) = -\frac{\mu^2}{2}\left[\pi^2 + (v+\sigma)^2\right] + \frac{\lambda}{4}\left[\pi^2 + (v+\sigma)^2\right]^2$$

$$= -\mu^2/2\,(\pi^2 + v^2 + 2v\sigma + \sigma^2)$$

$$+ \frac{\lambda}{4}(\pi^4 + 2\pi^2v^2 + 4\pi^2v\sigma + 2\pi^2\sigma^2$$

$$+ v^4 + 4v^3\sigma + 6v^2\sigma^2 + 4v\sigma^3 + \sigma^4)$$

now $v^2 = \mu^2/\lambda$ so e.g. the term quadratic in π is:

$$-\frac{\mu^2}{2}\pi^2 + \frac{\lambda}{4}\, 2\pi^2 v^2 = -\frac{\mu^2}{2}\pi^2 + \frac{\lambda}{2}\pi^2\frac{\mu^2}{\lambda}$$

$$= 0$$

so there is no π mass term!

$$V(\Phi^2) = -\left(\cancel{\frac{\mu^2}{2}\pi^2} + \frac{\mu^2}{2}v^2 + \cancel{\mu^2\frac{\mu}{\sqrt{\lambda}}\sigma} + \underline{\frac{\mu^2}{2}\sigma^2}\right)$$

$$+ \frac{\lambda}{4}\pi^4 + \cancel{\frac{\lambda}{2}\pi^2\frac{\mu^2}{\lambda}} + \lambda\pi^2\sqrt{\frac{\mu^2}{\lambda}}\,\sigma + \frac{\lambda}{2}\pi^2\sigma^2$$

$$+ \frac{\lambda v^4}{4} + \cancel{\lambda\frac{\mu^3}{\lambda^{3/2}}\sigma} + \underline{\frac{3\lambda}{2}\frac{\mu^2}{\lambda}\sigma^2}$$

$$+ \lambda\sqrt{\frac{\mu^2}{\lambda}}\,\sigma^3 + \frac{\lambda}{4}\sigma^4$$

$$= (\text{const}) + \frac{1}{2}(2\mu^2)\sigma^2 + \sqrt{\lambda}\mu\,\pi^2\sigma + \sqrt{\lambda}\mu\,\sigma^3$$

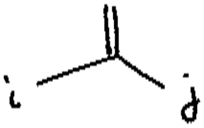
$$+ \frac{\lambda}{2}\pi^2\sigma^2 + \frac{\lambda}{4}\sigma^4 + \frac{\lambda}{4}(\pi^2)^2$$

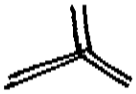
The σ has $(\text{mass})^2 = 2\mu^2$; the π has mass $= 0$

$\longleftarrow$ $\quad \overline{\sigma\,\sigma} = \dfrac{i}{p^2 - 2\mu^2}$

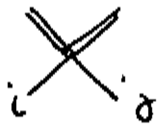
$i \longleftarrow j$ $\quad \overline{\pi^i\pi^j} = \delta^{ij}\,\dfrac{i}{p^2}$

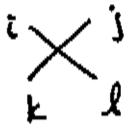
the vertices are:

$\pi^2\sigma \longrightarrow$  $= -2i\sqrt{\lambda}\mu\,\delta^{ij}$

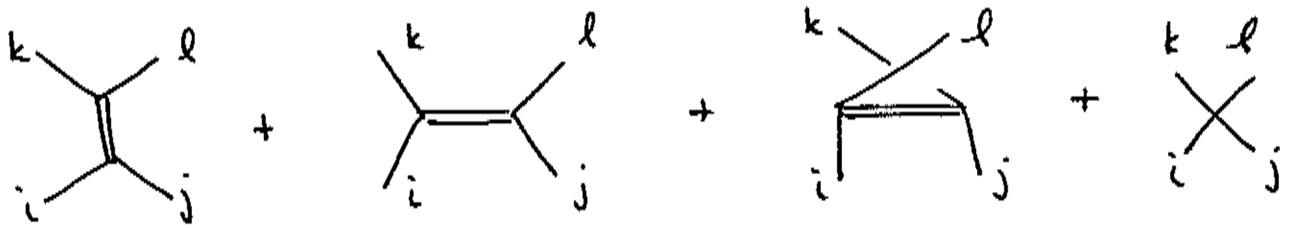
$\sigma^3 \longrightarrow$  $= -3!\,i\sqrt{\lambda}\mu = -6i\sqrt{\lambda}\mu$

$\sigma^4 \longrightarrow$  $= -6i\lambda$

$\pi^2\sigma^2 \longrightarrow$  $= -2i\,\delta^{ij}\lambda$

$(\pi^2)^2 \longrightarrow$  $= -2i(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})\lambda$

c) For $\pi^i(p_1) + \pi^j(p_2) \rightarrow \pi^k(p_3) + \pi^l(p_4)$



$$
\begin{aligned}
i\mathcal{M} = {} & (-2i\sqrt{\lambda}\mu)^2 \delta^{ij}\delta^{kl} \frac{i}{(p_1+p_2)^2 - 2\mu^2} \\
& + (-2i\sqrt{\lambda}\mu)^2 \delta^{ik}\delta^{jl} \frac{i}{(p_1-p_3)^2 - 2\mu^2} \\
& + (-2i\sqrt{\lambda}\mu)^2 \delta^{il}\delta^{jk} \frac{i}{(p_1-p_4)^2 - 2\mu^2} \\
& + (-2i\lambda)(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})
\end{aligned}
$$

then

$$
\begin{aligned}
i\mathcal{M} = \ & \delta^{ij}\delta^{kl}\left\{ -4i\lambda\mu^2 \frac{1}{(p_1+p_2)^2 - 2\mu^2} - 2i\lambda \right\} \\
& + \delta^{ik}\delta^{jl}\left\{ -4i\lambda\mu^2 \frac{1}{(p_1-p_3)^2 - 2\mu^2} - 2i\lambda \right\} \\
& + \delta^{il}\delta^{jk}\left\{ -4i\lambda\mu^2 \frac{1}{(p_1-p_4)^2 - 2\mu^2} - 2i\lambda \right\}
\end{aligned}
$$

If we set $p_1 = p_2 = p_3 = p_4 = 0$ (this is the threshold for these massless particles)

$$
\begin{aligned}
i\mathcal{M} = \ & \delta^{ij}\delta^{kl}\left\{ \frac{4i\lambda\mu^2}{2\mu^2} - 2i\lambda \right\} \\
& + \delta^{ik}\delta^{jl}\left\{ \frac{4i\lambda\mu^2}{2\mu^2} - 2i\lambda \right\} \\
& + \delta^{il}\delta^{jk}\left\{ \frac{4i\lambda\mu^2}{2\mu^2} - 2i\lambda \right\} = 0 \ !
\end{aligned}
$$

Expanding to one further power of momentum

$$
\begin{aligned}
\frac{1}{(p_1+p_2)^2 - 2\mu^2} &= -\frac{1}{2\mu^2}\left(1 - \frac{(p_1+p_2)^2}{2\mu^2}\right)^{-1} \\
&= -\frac{1}{2\mu^2} - \frac{(p_1+p_2)^2}{4\mu^4} + \cdots
\end{aligned}
$$

then

$$i\mathcal{M} \cong \delta^{ij}\delta^{kl}\left[-4i\lambda\mu^2\left(-\frac{(p_1+p_2)^2}{4\mu^4}\right)\right] + (2\text{ more})$$

$$i\mathcal{M} \cong \frac{i\lambda}{\mu^2}\left[\delta^{ij}\delta^{kl}(p_1+p_2)^2 + \delta^{ik}\delta^{jl}(p_1-p_3)^2 + \delta^{il}\delta^{jk}(p_1-p_4)^2\right]$$

$$+ \cdots$$

for $N=2$ $\quad ijkl = 1$ only. Now since $p_i^2 = 0$

$$(p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2$$

$$= 2p_1 p_2 - 2p_1 p_3 - 2p_1 p_4$$

$$= 2p_1(p_2 - p_3 - p_4) = 2p_1(-p_1) = -2p_1^2 = 0$$

so for $N=2$ $\quad i\mathcal{M} = \mathcal{O}(p^4)$.

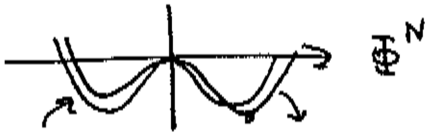
d.) Now add to V $\qquad \Delta V = -a\Phi^N$.

$$V = -\frac{\mu^2}{2}(\Phi^i)^2 + \frac{\lambda}{4}((\Phi^i)^2)^2 - a\Phi^N$$

$i<N$ $\quad \frac{\partial V}{\partial \Phi^i} = -\mu^2\Phi^i + \lambda\Phi^2\Phi^i$

$$\frac{\partial V}{\partial \Phi^N} = -\mu^2\Phi^N + \lambda\Phi^2\Phi^N - a$$

so the potential tilts toward Φ^N



the minimum is at $\Phi^i = 0 \quad i < N$

and Φ^N satisfying:

$$-\mu^2 \Phi^N + \lambda (\Phi^N)^3 - a = 0$$

write $\Phi^N = \frac{\mu}{\sqrt{\lambda}} + \delta \Phi^N$

$$0 = -\mu^2 \left(\cancel{\frac{\mu}{\sqrt{\lambda}}} + \delta\Phi^N\right) + \lambda\left[\cancel{\left(\frac{\mu}{\sqrt{\lambda}}\right)^3} + 3\frac{\mu^2}{\lambda}\delta\Phi^N + \cdots\right] - a$$

$$0 = -\mu^2 \delta\Phi^N + 3\mu^2 \delta\Phi^N - a$$

$$= 2\mu^2 \delta\Phi^N - a$$

or $\delta\Phi^N \cong \frac{a}{2\mu^2}$ $\qquad v = \frac{\mu}{\sqrt{\lambda}} + \frac{a}{2\mu^2} + \cdots$

Now find the term of order π^2 in V:

$$V = -\frac{\mu^2}{2}(\Phi^i)^2 + \frac{\lambda}{4}((\Phi^i)^2)^2 - a\Phi^N$$

$$\approx -\frac{\mu^2}{2}(\pi^i)^2 + \frac{\lambda}{2} v^2 \pi^2 \quad + \sigma^2 + \text{nonlinear}$$

$$V = -\frac{\mu^2}{2}\pi^2 + \frac{\lambda}{2}\left(\frac{\mu^2}{\lambda} + \frac{\mu}{\sqrt{\lambda}}\frac{a}{\mu^2} + \cdots\right)\pi^2$$

$$= \frac{\sqrt{\lambda}}{2}\,\frac{a}{\mu}\,\pi^2$$

so $$m_\pi^2 = \frac{\sqrt{\lambda}}{\mu}a + \cdots$$ for small a.