Physics 330 – Problem Set #4

Solutions

1.) Under P: $\quad \mathcal{P}\,\psi(t,\vec{x})\,\mathcal{P} = \gamma^0\psi(t,-\vec{x})$

then

$$\mathcal{P}\,\bar{\psi}\,\mathcal{P} = (\psi^\dagger\gamma^0)\gamma^0 = \bar{\psi}\gamma^0(t,-\vec{x})$$

then

$$\mathcal{P}\,\bar{\psi}\psi\,\mathcal{P} = \bar{\psi}\gamma^0\gamma^0\psi = +\bar{\psi}\psi \qquad (\gamma^0)^2 = 1$$

$$\mathcal{P}\,i\bar{\psi}\gamma^5\psi\,\mathcal{P} = i(\bar{\psi}\gamma^0)\gamma^5(\gamma^0\psi) = -i\bar{\psi}\gamma^5\psi \qquad \gamma^0\gamma^5 = -\gamma^5\gamma^0$$

$$\mathcal{P}\,\bar{\psi}\gamma^\mu\psi\,\mathcal{P} = \bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi$$

$$\mu = 0 \quad = \bar{\psi}(\gamma^0)^3\psi = +\bar{\psi}\gamma^0\psi$$

$$\mu = i \quad = \bar{\psi}\gamma^0\gamma^i\gamma^0\psi = -\bar{\psi}\gamma^i\psi \qquad \gamma^0\gamma^i = -\gamma^i\gamma^0$$

$$\mathcal{P}\,\bar{\psi}\gamma^\mu\gamma^5\psi\,\mathcal{P} = \bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi$$

$$\mu = 0 \quad = \bar{\psi}(\gamma^0)^2\gamma^5\gamma^0\psi = -\bar{\psi}\gamma^0\gamma^5\psi$$

$$\mu = i \quad = \bar{\psi}\gamma^0\gamma^i\gamma^5\gamma^0\psi = +\bar{\psi}\gamma^i\gamma^5\psi$$

γ^0 commutes w. $(\gamma^i\gamma^5)$

$$\mathcal{P}\,\bar{\psi}\sigma^{\mu\nu}\psi\,\mathcal{P} = \bar{\psi}\gamma^0\sigma^{\mu\nu}\gamma^0\psi$$

$\mu\nu = 0i$ $\quad$ γ^0 anticommutes w. $\gamma^0\gamma^i \Rightarrow \quad -\bar{\psi}\sigma^{\mu\nu}\psi$

$\mu\nu = ij$ $\quad$ γ^0 <u>commutes</u> w. $\gamma^i\gamma^j \Rightarrow \quad +\bar{\psi}\sigma^{\mu\nu}\psi$

so under P :

$$\left[\begin{array}{cccccc} & \bar\Psi\psi & i\bar\Psi\gamma^5\psi & \bar\Psi\gamma^\mu\psi & \bar\Psi\gamma^\mu\gamma^5\psi & \bar\Psi\sigma^{\mu\nu}\psi \\ P & +1 & -1 & (-1)^\mu & -(-1)^\mu & (-1)^\mu(-1)^\nu \end{array}\right.$$

where $(-1)^\mu = \begin{cases} +1 & \mu=0 \\ -1 & \mu=i \end{cases}$

Under T: $T\,\psi(t,\vec{x})\,T = -\gamma^1\gamma^3\,\psi(-t,\vec{x})$ $\qquad (\gamma^1\gamma^3)(-\gamma^1\gamma^3)=1$

$$T\,\bar\Psi(t,\vec{x})\,T = \psi^+(\gamma^3)^+(-\gamma^1)^+(\gamma^0)^* = +\bar\Psi\gamma^1\gamma^3\,(t,\vec{x})$$

then

$$T\bar\Psi\psi T = \bar\Psi(\gamma^1\gamma^3)(-\gamma^1\gamma^3)\psi = +\bar\Psi\psi$$

$$T(i\bar\Psi\gamma^5\psi)T = -i\,\bar\Psi\gamma^1\gamma^3\underbrace{(\gamma^5)^*}_{\gamma^5}(-\gamma^1\gamma^3)\psi = -i\,\bar\Psi\gamma^5\psi$$

$$T\bar\Psi\gamma^\mu\psi T = \bar\Psi\gamma^1\gamma^3(\gamma^\mu)^*(-\gamma^1\gamma^3)\psi$$

$\mu=0 \quad (\gamma^0)^* = +\gamma^0 \quad \gamma^0$ commutes w. $\gamma^1\gamma^3 \quad = \quad +\bar\Psi\gamma^0\psi$

$\mu=1,3 \quad (\gamma^i)^* = +\gamma^i \quad \gamma^i$ anticommutes w. $\gamma^1\gamma^3 \quad = \quad -\bar\Psi\gamma^i\psi$

$\mu=2 \quad (\gamma^2)^* = -\gamma^2 \quad \gamma^2$ commutes w. $\gamma^1\gamma^3 \quad = \quad -\bar\Psi\gamma^2\psi$

$$T\bar\Psi\gamma^\mu\gamma^5\psi T = \bar\Psi\gamma^1\gamma^3(\gamma^\mu)^*(\gamma^5)^*(-\gamma^1\gamma^3)\psi$$

$\mu=0 \quad (\gamma^0\gamma^5)^* = +\gamma^0\gamma^5 \quad \gamma^5$ commutes w. $\gamma^1\gamma^3 \quad = \quad +\bar\Psi\gamma^0\gamma^5\psi$

$\mu=1,3 \quad (\gamma^i\gamma^5)^* = +\gamma^i\gamma^5 \quad \to \quad -\bar\Psi\gamma^i\gamma^5\psi$

$\mu=2 \quad (\gamma^2\gamma^5)^* = -\gamma^2\gamma^5 \quad \to \quad -\bar\Psi\gamma^2\gamma^5\psi$

$$T\bar{\Psi}\sigma^{\mu\nu}\psi\, T = \bar{\Psi}\gamma^1\gamma^3(\sigma^{\mu\nu})^*(-\gamma^1\gamma^3)\psi$$

$(\sigma^{01})^* = \frac{-i}{2}[\gamma^0,\gamma^1] = -\sigma^{01}$ anticommutes w. $\gamma^1\gamma^3 \rightarrow +\bar{\Psi}\sigma^{01}\psi$

$\sigma^{03} \longrightarrow$ same story $\dashrightarrow$ $\rightarrow +\bar{\Psi}\sigma^{03}\psi$

$(\sigma^{02})^* = \frac{-i}{2}[\gamma^0,-\gamma^2] = +\sigma^{02}$ commutes w. $\gamma^1\gamma^3 \rightarrow +\bar{\Psi}\sigma^{02}\psi$

$(\sigma^{12})^* = \frac{-i}{2}[\gamma^1,-\gamma^2] = +\sigma^{12}$ anticommutes w. $\gamma^1\gamma^3 \rightarrow -\bar{\Psi}\sigma^{12}\psi$

$(\sigma^{32}) \longrightarrow$ same story $\rightarrow$ $\rightarrow -\bar{\Psi}\sigma^{32}\psi$

$(\sigma^{13})^* = \frac{-i}{2}[\gamma^1,\gamma^3] = -\sigma^{13}$ commutes w. $\gamma^1\gamma^3 \rightarrow -\bar{\Psi}\sigma^{13}\psi$

in all

	$\bar{\Psi}\psi$	$i\bar{\Psi}\gamma^5\psi$	$\bar{\Psi}\gamma^\mu\psi$	$\bar{\Psi}\gamma^\mu\gamma^5\psi$	$\bar{\Psi}\sigma^{\mu\nu}\psi$
T:	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$

Under C:

$$C\psi(x)C = -i(\bar{\Psi}\gamma^0\gamma^2)^T$$

$$C\bar{\Psi}C = (i\gamma^2\psi)^T\gamma^0 = -i(\gamma^0\gamma^2\psi)^T$$

then

$$C\bar{\Psi}\psi C = (-i)(\gamma^0\gamma^2\psi)^T(-i)(\bar{\Psi}\gamma^0\gamma^2)^T$$

$$= -(-i)^2\,\bar{\Psi}\gamma^0\gamma^2\gamma^0\gamma^2\psi \qquad \gamma^2\gamma^0\gamma^2 = -(\gamma^2)^2\gamma^0 = +\gamma^0$$

$$= +\bar{\Psi}\psi$$

$$Ci\bar{\Psi}\gamma^5\psi C = i(-i)^2(\gamma^0\gamma^2\psi)^T\gamma^5(\bar{\Psi}\gamma^0\gamma^2)^T$$

$$= (-1)(-i)\,\bar{\Psi}\gamma^0\gamma^2(\gamma^5)^T\gamma^0\gamma^2\psi$$

$$= +\bar{\Psi}\gamma^5\psi$$

in general

$$C\bar\Psi\Gamma\Psi C = (-1)(-i)^2\ \bar\Psi\gamma^0\gamma^2(\Gamma)^T\gamma^0\gamma^2\Psi$$

$$= +\bar\Psi\gamma^0\gamma^2(\Gamma)^T\gamma^0\gamma^2\Psi \quad \text{a.}\ (\gamma^0\gamma^2)^2 = +1$$

$$C\bar\Psi\gamma^\mu\Psi C = \bar\Psi\gamma^0\gamma^2(\gamma^\mu)^T\gamma^0\gamma^2\Psi$$

$\mu=0,2$ $\quad (\gamma^\mu)^T = +\gamma^\mu \quad \gamma^\mu$ anticommutes w. $\gamma^0\gamma^2 \quad = -\bar\Psi\gamma^\mu\Psi$

$\mu=1,3$ $\quad (\gamma^\mu)^T = -\gamma^\mu \quad \gamma^\mu$ commutes w. $\gamma^0\gamma^2 \quad = -\bar\Psi\gamma^\mu\Psi$

$$C\bar\Psi\gamma^\mu\gamma^5\Psi C = \bar\Psi\gamma^0\gamma^2(\gamma^\mu\gamma^5)^T\gamma^0\gamma^2\Psi$$

$(\gamma^5)^T = +\gamma^5$

$$= \bar\Psi\gamma^0\gamma^2\ \gamma^5(\gamma^\mu)^T\gamma^0\gamma^2\Psi$$

$$= \bar\Psi\gamma^0\gamma^2(\gamma^\mu)^T\gamma^0\gamma^2\gamma^5\times(-1)^3\ \Psi$$

$$= -(-1)^3\ \bar\Psi\gamma^\mu\gamma^5\Psi = +\bar\Psi\gamma^\mu\gamma^5\Psi$$

$$C\bar\Psi\sigma^{\mu\nu}\Psi C = \bar\Psi\gamma^0\gamma^2(\sigma^{\mu\nu})^T\gamma^0\gamma^2\Psi$$

$\mu=01$ $\quad (\sigma^{01})^T = (i\gamma^0\gamma^1)^T = i(-\gamma^1\gamma^0) = +\sigma^{01}$; this anticommutes w. $\gamma^0\gamma^2$

$$= -\bar\Psi\sigma^{01}\Psi$$

$\mu=03$ is the same story

$\mu=02$ $\quad (\sigma^{02})^T = (i\gamma^0\gamma^2)^T = i\gamma^2\gamma^0 = -\sigma^{02}$; this commutes w. $\gamma^0\gamma^2$

$$= -\bar\Psi\sigma^{02}\Psi$$

$\mu=13$ $\quad (\sigma^{13})^T = (i\gamma^1\gamma^3)^T = (i\gamma^3\gamma^1) = -\sigma^{13}$; this commutes w. $\gamma^0\gamma^2$

$$= -\bar\Psi\sigma^{13}\Psi$$

$\mu=12$ $\quad (\sigma^{12})^T = (i\gamma^1\gamma^2)^T = [i\gamma^2(-\gamma^1)] = +\sigma^{12}$; this anticommutes w. $\gamma^0\gamma^2$

$$= -\bar\Psi\sigma^{12}\Psi$$

$\mu=32$ $\quad \sigma^{32}$ is the same story, so in all

$$C\bar\Psi\sigma^{\mu\nu}\Psi C = -\bar\Psi\sigma^{\mu\nu}\Psi$$

in all

$$\left[\begin{array}{cccccc} & \bar\Psi\psi & i\bar\Psi\gamma^5\psi & \bar\Psi\gamma^\mu\psi & \bar\Psi\gamma^\mu\gamma^5\psi & \bar\Psi\sigma^{\mu\nu}\psi \\ C & +1 & -1 & -1 & +1 & -1 \end{array}\right.$$

2.) a.) $\langle \overset{s't'}{\Phi}(K) | \overset{st}{\Phi}(P) \rangle$

$$= \sqrt{2E_K 2E_P} \int \frac{d^3k}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \tilde\phi^*(\vec k)\,\tilde\phi(\vec p)$$

$$\langle 0 | b^{t'}_{K/2-k}\, a^{s'}_{K/2+k} \quad a^{s\,+}_{P/2+p}\, b^{t\,+}_{P/2-p} |0\rangle$$

the matrix element is:

$$(2\pi)^3 \delta^{(3)}(k-p + K/2 - P/2)\,\delta^{ss'}\,(2\pi)^3\,\delta^{(3)}(k-p - K/2 + P/2)\,\delta^{tt'}$$

$$= (2\pi)^3 \delta^{(3)}(k-p - \tfrac{1}{2}(K-P))\,(2\pi)^3\,\delta^{(3)}(-\vec K + \vec P)\,\delta^{ss'}\delta^{tt'}$$

$$= (2\pi)^3\,\delta(\overrightarrow{K-P})\,(2\pi)^3\,\delta(\vec k - \vec p)\,\delta^{ss'}\delta^{tt'}$$

so $\langle \overset{s't'}{\Phi}(K) | \overset{st}{\Phi}(P) \rangle = \underbrace{2E_P\,(2\pi)^3\delta(\overrightarrow{K-P})}_{\text{this is our conventional relativistic normalization}}\,\delta^{ss'}\delta^{tt'} \underbrace{\int \frac{d^3p}{(2\pi)^3} |\tilde\phi(\vec p)|^2}_{=1}$

For a 2-body bound state

$\vec r_1 = \vec R + \vec r/2 \qquad \vec r_2 = \vec R - \vec r/2$

the conjugate momenta are.

$\vec K = k_1 + k_2 \qquad \vec k = \frac{k_1 - k_2}{2}$

CM $\qquad$ internal

since then $[R^i, K^j] = [\frac{r_1^i + r_2^i}{2}, k_1^j + k_2^j] = i\delta^{ij}$

$$[r^i, k^j] = [r_1^i - r_2^i, \frac{k_1^j - k_2^j}{2}] = i\delta^{ij}$$

and $[\vec{R}, \vec{k}] = [\vec{r}, \vec{K}] = 0$

$\vec{K}$ and $\vec{k}$ are the CM and internal momenta of the wavefunction, so the Fourier transform of $\tilde{\Phi}(\vec{k})$:

$$\phi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \tilde{\phi}(\vec{k})$$

is the normal Schrödinger wavefunction of the bound state.

b.) We need the part of $\bar{\Psi}\Gamma\Psi(0)$ proportional to $b\,a$.

This is

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \; b_p^{t'} \, \bar{v}^{t'}(p) \, \Gamma \, u^{s'}(q) \; a_q^{s'}$$

and for $\vec{K} = 0$

$$\langle 0| b_p^{t'} a_q^{s'} \; a_k^{+s} b_{-k}^{+t} |0\rangle = (2\pi)^3 \delta^{(3)}(\vec{q} - \vec{k}) (2\pi)^3 \delta^{(3)}(\vec{p} + \vec{k}) \delta^{ss'} \delta^{tt'}$$

so

$$\langle 0| \bar{\Psi}\Gamma\Psi(0) | \Phi^{st}(K=0)\rangle = \int \frac{d^3k \sqrt{2E_k}}{(2\pi)^3} \frac{1}{2E_k} \tilde{\phi}(\vec{k}) \; \bar{v}^t(-\vec{k}) \, \Gamma \, u^s(\vec{k})$$

now use:

$$u^s(k) = \begin{pmatrix} \sqrt{k\cdot\sigma}\, \xi^s \\ \sqrt{k\cdot\bar{\sigma}}\, \xi^s \end{pmatrix}$$

$$v^t(-k) = \begin{pmatrix} \sqrt{\tilde{k}\cdot\sigma}\, \xi^{-t} \\ -\sqrt{\tilde{k}\cdot\bar{\sigma}}\, \xi^{-t} \end{pmatrix} \qquad \bar{v}(-k) = (-\xi^{-t\dagger}\sqrt{k\cdot\sigma}, \; \xi^{-t\dagger}\sqrt{k\cdot\bar{\sigma}})$$

$$\tilde{k}\cdot\sigma = k\cdot\bar{\sigma}$$

so $\quad \bar{v}^\dagger(-\vec{k})\,\gamma^\mu\, u(\vec{k}) = \bar{v}^\dagger(-\vec{k}) \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} u(\vec{k})$

$$= \xi^{-t}\left[-\sqrt{k\cdot\sigma}\,\sigma^\mu\sqrt{k\cdot\bar{\sigma}} + \sqrt{k\cdot\bar{\sigma}}\,\bar{\sigma}^\mu\sqrt{k\cdot\sigma}\right]\xi^s$$

In the non relativistic approximation $\quad E_k \approx m \gg |\vec{k}|$

$$\sqrt{k\cdot\sigma} \approx \sqrt{k\cdot\bar{\sigma}} \approx \sqrt{m}$$

$$\bar{v}^\dagger(-\vec{k})\,\gamma^\mu\, u(\vec{k}) = (-m)\;\xi^{-t}\,(0, 2\vec{\sigma})^\mu\,\xi^s$$

similarly $\quad \bar{v}^\dagger(-\vec{k})\,\gamma^\mu\gamma^5\, u(k) = \bar{v}^\dagger(-k)\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} u(k)$

$$= \xi^{-t\dagger}\left[-\sqrt{k\cdot\sigma}\,\sigma^\mu\sqrt{k\cdot\bar{\sigma}} - \sqrt{k\cdot\bar{\sigma}}\,\bar{\sigma}^\mu\sqrt{k\cdot\sigma}\right]\xi^s$$

in the NR limit $= -m\;\xi^{-t\dagger}\left[(2, \vec{0})^\mu\right]\xi^s$

Now for $\quad \xi^s \qquad \xi^{-s} = (-i\sigma^2)\,\xi^{s*} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\xi^{s*}$

$s = \uparrow \quad \xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^{-s} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$s = \downarrow \quad \xi^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \xi^{-s} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

in this part, we want $\quad \xi^{-t\dagger} = (0\;1) \qquad \xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

so $\quad \sigma^1\ \sigma^2$ have nonzero matrix elements, all other matrix elements are zero in the NR limit.

$$\xi^{-t\dagger}\,\sigma^i\,\xi^s = (1, i, 0)^i$$

$$\xi^{-t\dagger}\,1\,\xi^s = 0$$

Putting all of the pieces together:

$$\langle 0|\ \bar\Psi\gamma^1\Psi\ |\vec{\Phi}^{\uparrow\uparrow}(k=0)\rangle = \underbrace{\sqrt{2E_K}}_{=\sqrt{2M_\Phi}}\ \underbrace{\int\frac{d^3k}{(2\pi)^3}\tilde{\Phi}(k)}_{\phi(0)}\ \underbrace{\left(\frac{-2m}{2E_k}\right)}_{=1\text{ in the NR approx.}}\cdot 1$$

the Schrödinger wavefunction at $\vec{r}=0$

again

$$\langle 0|\ \bar\Psi\gamma^1\Psi(0)\ |\Phi^{\uparrow\uparrow}(k=0)\rangle = -\sqrt{2M_\Phi}\ \ \phi(0)\cdot 1$$

$$\langle 0|\ \bar\Psi\gamma^2\Psi(0)\ |\Phi^{\uparrow\uparrow}(k=0)\rangle = -\sqrt{2M_\Phi}\ \ \phi(0)\cdot i$$

in other words.

$$\langle 0|\ \bar\Psi\gamma^\mu\Psi(0)\ |\Phi^{\uparrow\uparrow}(k=0)\rangle = -\sqrt{2M_\Phi}\ \phi(0)\sqrt{2}\,\varepsilon^\mu$$

where $\varepsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0)^\mu$

is a polarization vector for $J^3 = +1$ $J=1$

The appearance of $\phi(0)$ also makes sense: the fermion and antifermion must come to $\vec{r}=0$ to be annihilated by $\bar\Psi\gamma^\mu\Psi(0)$

also $\langle 0|\ \bar\Psi\gamma^\mu\gamma^5\Psi(0)\ |\Phi^{\uparrow\uparrow}(k=0)\rangle = 0$

this makes sense, because $|\Phi^{\uparrow\uparrow}\rangle$ has

$$P = -1 \quad \text{and} \quad C = -1 \qquad C a^\dagger b^\dagger |0\rangle = b^\dagger a^\dagger |0\rangle = -a^\dagger b^\dagger |0\rangle$$

$\bar\Psi \vec\gamma \Psi$ has $P = -1$, $C = -1$

$\bar\Psi \vec\gamma \gamma^5 \Psi$ has $P = +1$, $C = +1$

(c.) Similarly, in the NR limit

$$\langle 0| \bar\Psi\Psi(0) |\Phi^{st}(k=0)\rangle = \sqrt{2M_\Phi}\, \phi(0) \frac{1}{2m}\, \bar v^t(\vec 0)\, u^s(\vec 0)$$

$$\langle 0| \bar\Psi\gamma^5\Psi(0) |\Phi^{st}(k=0)\rangle = \sqrt{2M_\Phi}\, \phi(0) \frac{1}{2m}\, \bar v^t(0)\gamma^5 u^s(0)$$

$$\bar v^t(\vec k)\, u^s(\vec k) = \xi^{-t\,\dagger} \left[-\sqrt{k\cdot\sigma}\sqrt{k\cdot\sigma} + \sqrt{k\cdot\bar\sigma}\sqrt{k\cdot\bar\sigma} \right] \xi^s$$

$$\cong 0 \quad \text{in the NR approx.}$$ (or when integrated over the direction of $\vec k$)

$$\bar v^t \gamma^5 u^s = \xi^{-t\,\dagger} \left[-\sqrt{k\sigma}\sqrt{k\sigma} - \sqrt{k\cdot\bar\sigma}\sqrt{k\bar\sigma} \right] \xi^s$$

$$= -2m \left(\xi^{-t\,\dagger} \xi^s \right)$$

for $s = \uparrow \; t = \downarrow \quad = -2m \, (-1\;0) \begin{pmatrix}1\\0\end{pmatrix} = +2m$

$s = \downarrow \; t = \uparrow \quad = -2m \, (0\;1) \begin{pmatrix}0\\1\end{pmatrix} = -2m$

so for $\quad \frac{1}{\sqrt2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \qquad S = 0$

$$\langle 0| \bar\Psi\Psi(0) |\Phi\rangle = 0$$

$$\langle 0| \bar\Psi\gamma^5\Psi(0) |\Phi\rangle = \sqrt{2M_\Phi}\, \phi(0) (-\sqrt2)$$

This makes sense, Because Φ will spin 0 has

$$P = -1 \qquad C = +1$$

check C: $C \frac{1}{\sqrt{2}}(a^{\uparrow +} b^{\downarrow +} - a^{\downarrow +} b^{\uparrow +})|0\rangle = \frac{1}{\sqrt{2}}(b^{\uparrow +} a^{\downarrow +} - b^{\downarrow +} a^{\uparrow +})|0\rangle$

$$= + \frac{1}{\sqrt{2}}(a^{\uparrow +} b^{\downarrow +} - a^{\downarrow +} b^{\uparrow +})|0\rangle$$

$\bar{\Psi}\psi$ has $P = +1 \quad C = +1$

$\bar{\Psi}\gamma^5\psi$ has $P = -1 \quad C = +1$

3.) a) $$\delta S = \int d^4x \left\{ -\frac{1}{4} \cdot 2 \cdot F^{\mu\nu} \cdot (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) - \delta A_\mu j^\mu \right\}$$

$$= \int d^4x \left\{ + \delta A_\nu \, \partial_\mu F^{\mu\nu} - \delta A_\nu j^\nu \right\}$$

$$= \int d^4x \; \delta A_\nu \left(\partial_\mu F^{\mu\nu} - j^\nu \right)$$

so $\qquad \partial_\mu F^{\mu\nu} = j^\nu$

b.) $\nu = 0 \qquad \partial_i F^{i0} = j^0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{E} = \rho$

$\nu = i \qquad \partial_0 F^{0i} + \partial_j F^{ji} = j^i$

$$-\frac{\partial E^i}{\partial t} + \partial_j \epsilon^{ijk} B^k = j^i$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j}$$

Where are the other two Maxwell equations?

writing $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ gives

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{E} = -\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\phi$$

from which $\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ are automatic.

the relativistic version of these equations is the identity

$$\epsilon^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = 0$$

c.)
$$S = \int d^4x \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \cdots \right)$$

$$= \int d^4x \quad \tfrac{1}{2} F^{0i} F^{0i} + \cdots$$

$$= \int d^4x \quad \tfrac{1}{2} \left(\dot{A}^i + \nabla^i \phi \right)^2 + \cdots$$

so
$$\frac{\delta \mathcal{L}}{\delta \dot{A}^i} = \dot{A}^i + \nabla^i \phi = -E^i$$

$$\frac{\delta \mathcal{L}}{\delta \dot{A}^0} = 0$$

d.)
$$\vec{\nabla} \cdot \vec{E} = \rho \Rightarrow -\nabla^2 A^0 - \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = \rho$$

so if $\vec{\nabla} \cdot \vec{A} = 0 \qquad -\nabla^2 A^0 = \rho = j^0$

then
$$A^0(\vec{x}) = \int d^3y \, \frac{1}{4\pi|\vec{x}-\vec{y}|} \, j^0(y)$$

e.) try
$$A^i(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p}} \, e^{i\vec{p}\cdot\vec{x}} \sum_a \left(\varepsilon_a^i(\vec{p})\, a_{pa} + \varepsilon_a^{i*}(-\vec{p})\, a^\dagger_{-\vec{p}a} \right)$$
$$\pi^i(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p}} (-ip)\, e^{i\vec{p}\cdot\vec{x}} \sum_a \left(\varepsilon_a^i(\vec{p})\, a_{pa} - \varepsilon_a^{i*}(-\vec{p})\, a^\dagger_{-\vec{p}a} \right)$$

$$[A^i(\vec{k}), \pi^j(-\vec{k}')] = \frac{1}{\sqrt{2k\, 2k'}} \sum_{a,b} (-ik')$$
$$\left[\varepsilon_a^i(\vec{k})\, a_{ka} + \varepsilon_a^{i*}(-\vec{k})\, a^\dagger_{-\vec{k}a} ,\; \varepsilon_b^j(-\vec{k}')\, a_{-\vec{k}'b} - \varepsilon_b^{j*}(\vec{k}')\, a^\dagger_{\vec{k}'b} \right]$$
$$= \frac{-i k'}{\sqrt{4kk'}} \sum_{a,b} \left(-\varepsilon_a^i(\vec{k})\, \varepsilon_b^{j*}(\vec{k}')\, (2\pi)^3 \delta(\vec{k}-\vec{k}')\, \delta_{ab} - \varepsilon_a^{i*}(-\vec{k})\, \varepsilon_b^j(-\vec{k}')(2\pi)^3 \delta(\vec{k}-\vec{k}')\, \delta_{ab} \right)$$
$$= i\, \frac{1}{2} \underbrace{\sum_{a=1,2} \left[\varepsilon_a^i(\vec{k})\, \varepsilon_a^{j*}(\vec{k}) + \varepsilon_a^{i*}(-\vec{k})\, \varepsilon_a^j(-\vec{k}) \right]}_{\text{this is the projector onto transverse directions}} (2\pi)^3 \delta(\vec{k}-\vec{k}')$$
$$= i \left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} \right) (2\pi)^3 \delta(\vec{k}-\vec{k}')$$

a similar calculation gives $[A^i(\vec{k}), A^j(-\vec{k}')] = [\pi^i(\vec{k}), \pi^j(-\vec{k}')] = 0$

f.) $\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}$

$\vec{\nabla}\cdot\vec{E} = -\nabla^2\phi$; the transverse part of $\vec{E}$ is built from $\vec{\pi}$. Then:

$$\vec{E} = \underbrace{\int d^3y \frac{(\vec{x}-\vec{y})}{4\pi|x-y|^2} j^0(y)}_{\text{a c-number}} - \vec{\pi}$$

$$\vec{E}(k) = -i\vec{k}\,\frac{+1}{|\vec{k}|^2}\, j^0(k) - \vec{\pi}(k)$$

and since $\vec{k}\cdot\vec{\pi}(k) = 0$

$$\int d^3x \;\tfrac{1}{2}E^2 = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2}\,\vec{E}(\vec{k})\cdot\vec{E}(-\vec{k}) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{2}\left|\vec{E}(\vec{k})\right|^2$$

$$= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\left\{ j^0(-\vec{k})\,\frac{1}{k^2}\, j^0(k) + \vec{\pi}(\vec{k})\cdot\vec{\pi}(-k)\right\}$$

g) $\int d^3x \;\tfrac{1}{2}B^2 = \int d^3x \;\tfrac{1}{2}(\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\vec{A})$

$$= \int d^3x \;\epsilon^{ijk}\epsilon^{ilm}\,\nabla^j A^k\,\nabla^l A^m$$

$$= \int d^3x \;A^k(x)\left(-\nabla^2\delta^{km} + \nabla^k\underbrace{\nabla^m) A^m}_{=0}\right.$$

$$= \int d^3x \;A^k(x)(-\nabla^2)A^k(x)$$

or $$\int d^3x \; \tfrac{1}{2} B^2 = \int \frac{d^3k}{(2\pi)^3} \; \tfrac{1}{2} \vec{A}(\vec{k}) \; k^2 \; \vec{A}(-\vec{k})$$

then

$$\mathcal{H} = \int d^3x \; \tfrac{1}{2}(E^2 + B^2)$$

$$= \int \frac{d^3k}{(2\pi)^3} \left\{ \tfrac{1}{2} j^0(\vec{k}) \frac{1}{|\vec{k}|^2} j^0(-\vec{k}) + \tfrac{1}{2} \vec{\Pi}(-\vec{k}) \vec{\Pi}(k) + \tfrac{1}{2} k^2 \vec{A}(-\vec{k}) \cdot \vec{A}(\vec{k}) \right\}$$

$$= \int \frac{d^3k}{(2\pi)^3} \Big\{ \tfrac{1}{2} j^0(\vec{k}) \frac{1}{|\vec{k}|^2} j^0(-\vec{k})$$

$$+ \tfrac{1}{2} \frac{1}{2k} (-k^2) \left(\varepsilon^i_a(-\vec{k}) a_{-\vec{k}a} - \varepsilon^{i*}_a(\vec{k}) a^\dagger_{ka} \right) \left(\varepsilon^i_b(k) a_{ka} - \varepsilon^{*i}_b(\vec{k}) a^\dagger_{-kb} \right)$$

$$+ \tfrac{1}{2} \frac{1}{2k} (+k^2) \left(\varepsilon^i_a a_{-\vec{k}a} + \varepsilon^{i*}_a a^\dagger_{\vec{k}a} \right) \left(\varepsilon^i_b a_{ka} + \varepsilon^*_b a^\dagger_{-\vec{k}b} \right) \Big\}$$

the terms w. aa and $a^\dagger a^\dagger$ cancel. What remains is:

$$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \left[\tfrac{1}{2} j^0(\vec{k}) \frac{1}{|\vec{k}|^2} j^0(\vec{k}) + \frac{k}{4} \cdot \left\{ 2\varepsilon^i_a a_{ka} a^\dagger_{kb} \varepsilon^{*i}_b + 2\varepsilon^{i*}_a a^\dagger_{ka} a_{kb} \varepsilon^i_b \right\} \right]$$

now $\varepsilon^i_a \varepsilon^{*i}_b = \delta_{ab} \qquad a,b = 1,2$ so

$$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \left\{ \tfrac{1}{2} j^0(-\vec{k}) \frac{1}{|\vec{k}|^2} j^0(\vec{k}) + k \; a^\dagger_{ka} a_{ka} + (\text{const}) \right\}$$

How can we use this Hamiltonian? Write

$$\mathcal{H} = \mathcal{H}_0 + H_{int}$$

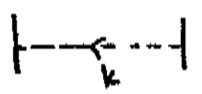
$$\mathcal{H}_0 = \int \frac{d^3k}{(2\pi)^3}\; k\, a^+_{ka} a_{ka}$$

$$H_{int} = \int \frac{d^3k}{(2\pi)^3}\; \frac{1}{2} j^0(-k) \frac{1}{|\vec{k}|^2} j^0(k) + \int \frac{d^3k}{(2\pi)^3} (-\vec{j}(-k) \cdot \vec{A}(k))$$

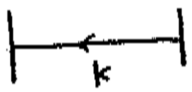
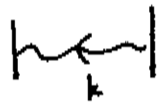
If we do perturbation theory in H_{int}, there are two vertices, a conventional one:

 $$= +i\, j^i(-k)$$

and a nonlocal instantaneous interaction

 $$= -i\, j^0(-k) \frac{1}{|\vec{k}|^2} j^0(k)$$

Put these together:

 + 

$$= -i\, j^0(-k) \frac{1}{|\vec{k}|^2} j^0(k) + i j^i(-k) \frac{i\left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2}\right)}{k^2 + i\varepsilon} (i j^j(k))$$

$$= -i\, j^0(-k) \frac{1}{|\vec{k}|^2} j^0(k) - i \frac{j^i(-k)\, j^i(k)}{k^2} + i \frac{\vec{k}\cdot\vec{j}(-k)\; \vec{k}\cdot\vec{j}(k)}{k^2 |\vec{k}|^2}$$

now use current conservation: $\partial_0 j^0 + \vec{\nabla}\cdot\vec{j} = 0$

$$\vec{k}\cdot\vec{j}(k) = k^0 j^0(k)$$

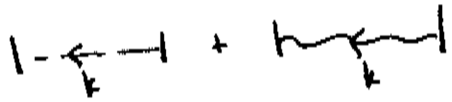
so

$$-i\, j^0(-k)\,\frac{1}{|\vec{k}|^2}\,j^0(k) \;+\; i\,\frac{\vec{k}\cdot\vec{j}(-k)\;\vec{k}\cdot\vec{j}(k)}{k^2\,|\vec{k}|^2}$$

$$= -i\, j^0(-k)\,\frac{\cancel{(k^0)^2}-|\vec{k}|^2}{k^2\,|\vec{k}|^2}\,j^0(k) \;+\; i\,\frac{\cancel{(k^0)^2}\,j^0(-k)\,j^0(k)}{k^2\,|\vec{k}|^2}$$

$$= i\,\frac{j^0(-k)\,j_0(k)}{k^2}$$

in all



$$= \left(i j^\mu(-k)\right)\Big(\underbrace{\frac{-i g_{\mu\nu}}{k^2+i\epsilon}}\Big)\left(i j^\nu(k)\right)$$

this is the photon propagator introduced in class.