Physics 330 - Problem Set #3

Solutions

1.) $u(p)$ satisfies $(\not{p}-m)\, u(p) = 0$ $\qquad \not{p} = \gamma \cdot p$

$\bar{u}(p')$ satisfies $\bar{u}(p')\,(\not{p}'-m) = 0$

so
$$\bar{u}(p')\,[\not{p}'\gamma^\mu + \gamma^\mu \not{p}\,]\,u(p)$$
$$= \bar{u}(p')\,(m\gamma^\mu + \gamma^\mu m)\,u(p) = 2m\,\bar{u}\gamma^\mu u$$

now
$$\not{p}'\gamma^\mu = \tfrac{1}{2}\left(\{\not{p}',\gamma^\mu\} + [\not{p}',\gamma^\mu]\right)$$
$$= p'^\mu - \tfrac{1}{2}[\gamma^\mu,\gamma^\nu]\,p'_\nu$$
$$= p'^\mu + i\,\sigma^{\mu\nu}p'_\nu$$

$$\gamma^\mu\not{p} = \tfrac{1}{2}\left(\{\gamma^\mu,\not{p}\} + [\gamma^\mu,\not{p}]\right)$$
$$= p^\mu + [\gamma^\mu,\gamma^\nu]\,p_\nu$$
$$= p^\mu - i\,\sigma^{\mu\nu}p_\nu$$

so
$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\,\bar{u}(p')\left\{(p'^\mu + p^\mu) + i\,\sigma^{\mu\nu}(p'_\nu - p_\nu)\right\}u(p)$$

as required

2.) a) χ transforms under Lorentz transformations as

$$\chi(x) \to \Lambda_L \chi(\Lambda^{-1}x)$$

where Λ_L is the top 2x2 matrix in $\Lambda_{1/2}$, with infinitesimal form

$$\Lambda_L = (1 - i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\eta}\cdot\vec{\sigma}/2)$$

Assume ~~that~~ $\chi(x)$ satisfies the Majorana equation

$$i\bar{\sigma}\cdot\partial\chi - im\sigma^2\chi^* = 0$$

We need to prove that $\chi'(x) = \Lambda_L\chi(\Lambda^{-1}x)$ satisfies this equation

$$i\bar{\sigma}\cdot\partial\, \Lambda_L\chi(\Lambda^{-1}x) - im\sigma^2\, \Lambda_L^*\, \chi^*(\Lambda^{-1}x)$$

$$= i\bar{\sigma}^\nu (\Lambda^{-1})^\mu{}_\nu\, \Lambda_L(\partial_\mu\chi)(\Lambda^{-1}x) - im\,(\sigma^2\Lambda_L^*)\,\chi^*(\Lambda^{-1}x)$$

in class we proved

$$\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\lambda\gamma^\lambda$$

in 2x2 blocks, this is

$$\begin{pmatrix}\Lambda_L^{-1} & \\ & \Lambda_R^{-1}\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}\Lambda_L & \\ & \Lambda_R\end{pmatrix} = \Lambda^\mu{}_\lambda\begin{pmatrix} & \sigma^\lambda \\ \bar{\sigma}^\lambda & \end{pmatrix}$$

or

$$\Lambda_L^{-1}\sigma^\mu\Lambda_R = \Lambda^\mu{}_\lambda\sigma^\lambda$$

$$\Lambda_R^{-1}\bar{\sigma}^\mu\Lambda_L = \Lambda^\mu{}_\lambda\bar{\sigma}^\lambda$$

so $$\bar\sigma^\nu \Lambda_L = \Lambda_R \Lambda_R^{-1} \bar\sigma^\nu \Lambda_L$$
$$= \Lambda_R \, \bar\sigma^\lambda \, \Lambda^\nu{}_\lambda$$

Now look at $\sigma^2 \Lambda_L^*$. The infinitesimal form is

$$\sigma^2 \Lambda_L^* \cong \sigma^2 \left(1 + i \vec\theta \cdot \frac{\vec\sigma^*}{2} - \vec\eta \cdot \frac{\vec\sigma^*}{2}\right)$$
$$= \left(1 - i \vec\theta \vec\sigma/2 + \vec\eta \vec\sigma/2\right) \sigma^2 \equiv \Lambda_R \sigma^2$$

taking repeated small steps, the same is true for finite transformations:

$$\sigma^2 \Lambda_L^* = \Lambda_R \sigma^2$$

then the above equation is:

$$\left[\Lambda_R \, i \bar\sigma^\lambda \Lambda^\nu{}_\lambda (\Lambda^{-1})^\mu{}_\nu \, \partial_\mu \chi - i m \, \Lambda_R \sigma^2 \chi^*\right](\Lambda^{-1} x)$$

$$\Lambda_R \left(i \bar\sigma \cdot \partial \chi - i m \sigma^2 \chi^*\right)(\Lambda^{-1} x) = 0$$

if χ satisfies the original equation.

So the Majorana equation is covariant. Now, if χ satisfies this equation

$$i \bar\sigma \cdot \partial \chi = i m \sigma^2 \chi^*$$

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$$-i \bar\sigma^* \partial \chi^* = i m \sigma^2 \chi$$

$$i\sigma\cdot\partial\, i\bar{\sigma}\cdot\partial\,\chi \quad = \quad -\partial^2\chi$$

$$= i\sigma\cdot\partial \;\; i m \sigma^2 \chi^*$$

$$= (i)^2 m \sigma^2 (\bar{\sigma}\cdot\partial)^* \chi^*$$

$$= -m\sigma^2(-m\sigma^2\chi) \quad = \quad m^2\chi$$

so $\quad (-\partial^2 - m^2)\chi = 0 \quad$ if χ satisfies the Majorana eq.

b.) $$S = \int d^4x \left\{ \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right\}$$

$$S^* = \int d^4x \left\{ \partial_\mu \chi^\dagger(-i\bar{\sigma}^\mu)\chi - i\frac{m}{2}\left(\chi^\dagger(\sigma^2)^\dagger\chi^* - \chi^T(\sigma^2)^\dagger\chi\right)\right\}$$

$$(\sigma^2)^\dagger = \sigma^2$$

$$= \int d^4x \left\{ \chi^\dagger i\bar{\sigma}\cdot\partial\chi - i\frac{m}{2}\left(\chi^\dagger\sigma^2\chi^* - \chi^T\sigma^2\chi\right)\right\}$$

$$= S$$

Now vary S with respect to χ, $\chi^\dagger$

$$\delta S = \int d^4x \, \{ \delta\chi^\dagger i\bar\sigma\cdot\partial\chi + \chi^\dagger i\bar\sigma\cdot\partial\,\delta\chi$$
$$+ \tfrac{im}{2} [\delta\chi^T\sigma^2\chi + \chi^T\sigma^2\delta\chi - \delta\chi^\dagger\sigma^2\chi^* - \chi^\dagger\sigma^2\delta\chi^*] \}$$

now

$$\delta\chi^T\sigma^2\chi = \chi^T(\sigma^2)^T\delta\chi \underset{\text{Grassmann}}{(-1)} = +\chi^T\sigma^2\delta\chi$$
$$\chi^\dagger\sigma^2\delta\chi^* = \delta\chi^\dagger(\sigma^2)^T\chi^*(-1) = +\delta\chi^\dagger\sigma^2\chi^*$$

so this organizes itself into:

$$\delta S = \int d^4x \, \{ \delta\chi^\dagger \{ i\bar\sigma\cdot\partial\chi - im\sigma^2\chi^* \}$$
$$+ \{ -i\partial_\mu\chi^\dagger\bar\sigma^\mu + im\chi^T\sigma^2 \} \delta\chi \}$$

then

$$\delta S = 0 \text{ implies } \begin{cases} i\bar\sigma\cdot\partial\chi - im\sigma^2\chi^* = 0 \\ -i\partial_\mu\chi^\dagger\bar\sigma^\mu + im\chi^T\sigma^2 = 0 \end{cases}$$

(which are equivalent, conjugate equations)

c.) Write the Dirac field as

$$\psi = \begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix}$$

then $\psi^\dagger = (\chi_1^\dagger, -i\chi_2^T\sigma^2)$ $\qquad \bar\psi = (-i\chi_2^T\sigma^2, \chi_1^\dagger)$

then

$$\mathcal{L} = \bar{\Psi}(i\gamma\cdot\partial - m)\psi$$

$$= (-i\chi_2^T\sigma^2,\ \chi_1^\dagger)\left\{\begin{pmatrix} 0 & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & 0\end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m\end{pmatrix}\right\}\begin{pmatrix}\chi_1 \\ i\sigma^2\chi_2^*\end{pmatrix}$$

$$= -i\chi_2^T\sigma^2(i\sigma\cdot\partial)\, i\sigma^2\chi_2^* + \chi_1^\dagger i\bar{\sigma}\cdot\partial\chi_1$$

$$+ im\,\chi_2^T\sigma^2\chi_1 - im\chi_1^\dagger\sigma^2\chi_2^*$$

now $\quad \sigma^2\ i\sigma\cdot\partial\ \sigma^2 = +i\ \bar{\sigma}^*\cdot\partial = +i\,(\bar{\sigma}^\mu)^T\partial_\mu$

~~integrate~~ by parts

$$-i\chi_2^T\sigma^2\ i\sigma\cdot\partial\ i\sigma^2\chi_2^* = \chi_2^T\ i(\bar{\sigma}^\mu)^T\partial_\mu\,\chi_2^*$$

$$= -i\ \partial_\mu\chi_2^T(\bar{\sigma}^\mu)^T\chi_2^*$$

$$= +i\ \chi_2^\dagger\bar{\sigma}\cdot\partial\chi_2$$

interchange $\chi_2\ \chi_2^*$ w. (-1) from Grassmannian

so!

$$\mathcal{L} = \chi_1^\dagger i\bar{\sigma}\cdot\partial\chi_1 + \chi_2^\dagger i\bar{\sigma}\cdot\partial\chi_2$$

$$+ i\frac{m}{2}\left(2\chi_1^T\sigma^2\chi_2 - 2\,\chi_1^\dagger\sigma^2\chi_2^*\right)$$

~~or~~ $\Rightarrow$

$$\mathcal{L} = \chi_1^\dagger i\bar\sigma\cdot\partial\chi_1 + \chi_2^\dagger i\bar\sigma\cdot\partial\chi_2 + i\tfrac{1}{2}\left(\chi_a^T\sigma^2 m_{ab}\chi_b - \chi_a^\dagger\sigma^2 m_{ab}\chi_b^*\right)$$

where $m_{ab} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$

d.) This Lagrangian has a symmetry

$$\chi_1 \to e^{i\alpha}\chi_1 \qquad \chi_2 \to e^{-i\alpha}\chi_2$$

$$\chi_1^\dagger \to \chi_1^\dagger e^{-i\alpha} \qquad \chi_2^\dagger \to \chi_2^\dagger e^{+i\alpha}$$

the corresponding current is

$$J^\mu = \chi_1^\dagger\bar\sigma^\mu\chi_1 - \chi_2^\dagger\bar\sigma^\mu\chi_2$$

and this should be conserved:

$$\partial_\mu J^\mu = \partial_\mu(\chi_1^\dagger\bar\sigma^\mu\chi_1) - \partial_\mu\chi_2^\dagger\bar\sigma^\mu\chi_2$$

the equations of motion are:

$$i\bar\sigma\cdot\partial\chi_1 - i\sigma^2 m\,\chi_2^* = 0$$

$$-i\partial_\mu\chi_1^\dagger\bar\sigma^\mu + im\,\chi_2^T\sigma^2 = 0$$

$$i\bar\sigma\cdot\partial\chi_2 - i\sigma^2 m\,\chi_1^* = 0$$

$$-i\partial_\mu\chi_2^\dagger\bar\sigma^\mu + im\,\chi_1^T\sigma^2 = 0$$

so $$\partial_\mu J^\mu = (\partial_\mu \chi_1^\dagger \bar\sigma^\mu)\,\chi_1 + \chi_1^\dagger(\bar\sigma\cdot\partial\chi_1)$$
$$- (\partial_\mu\chi_2^\dagger\bar\sigma^\mu)\,\chi_2 - \chi_2^\dagger(\bar\sigma\cdot\partial\chi_2)$$
$$= m\,\chi_2^T\sigma^2\chi_1 + \chi_1^\dagger\, m\sigma^2\chi_2^*$$
$$- m\chi_1^T\sigma^2\chi_2 - \chi_2^\dagger\, m\sigma^2\chi_1^*$$
$$= 0 \qquad \text{since} \quad \chi_2^T\sigma^2\chi_1 = \chi_1^T\sigma^2\chi_2 \quad \text{etc.}$$

actually $\partial_\mu J^\mu = \partial_\mu(\bar\Psi\gamma^\mu\Psi)$ in the Dirac theory.

On the other hand

$$\partial_\mu(\chi_1^\dagger\bar\sigma^\mu\chi_1 + \chi_2^\dagger\bar\sigma^\mu\chi_2)$$
$$= 2m\,(\chi_1^T\sigma^2\chi_2 - \chi_1^\dagger\sigma^2\chi_2^*)$$

this current generates $\chi_1 \to e^{i\alpha}\chi_1 \quad \chi_2 \to e^{i\alpha}\chi_2$
which is not a symmetry for $m \neq 0$.

The comparison to the Dirac theory is

$$\chi_1^\dagger\bar\sigma^\mu\chi_1 - \chi_2^\dagger\bar\sigma^\mu\chi_2 = \bar\Psi\gamma^\mu\Psi$$
$$\chi_1^\dagger\bar\sigma^\mu\chi_1 + \chi_2^\dagger\bar\sigma^\mu\chi_2 = -\bar\Psi\gamma^\mu\gamma^5\Psi$$

Now consider a thy of N 2-component fermions:

$$\mathcal{L} = \chi_a^\dagger i\bar\sigma\cdot\partial\chi_a + i\frac{m}{2}\left(\chi_a^T\sigma^2\chi_a - \chi_a^\dagger\sigma^2\chi_a\right)$$

summed over $a = 1 \longrightarrow N$

If R_{ab} is a rotation in N-dim space $\quad x \to Rx$

(elements of R real) $\quad x\cdot y \to x R^T R y = x\cdot y$

so $\quad R^T R = 1$

$$\mathcal{L} \to \chi^\dagger R^T i\bar\sigma\cdot\partial R\chi + i\frac{m}{2}\left(\chi^T R^T\sigma^2 R\chi - \chi^\dagger R^T\sigma^2 R\chi\right)$$

$R^T R = 1$ so

$$= \mathcal{L}$$

the kinetic term $\chi_a^\dagger i\bar\sigma\cdot\partial\chi_a$ is invariant to

unitary transformations $\quad \chi \to U\chi \quad U \in U(N)$

$N\times N$ unitary

the full $\mathcal{L}$ is invariant to rotations

$$\chi \to R\chi \qquad R \in O(N)$$

e.) Follow the method in the problem set. First, check that the Hamiltonian given leads to the Majorana equation as its equation of motion:

$$i\frac{\partial}{\partial t}\chi(\vec{x}) = [\chi(\vec{x}), H]$$

$$= \left[\chi(\vec{x}), \int d^3y \left(\chi^\dagger (i\vec{\sigma}\vec{\nabla})\chi(y) + i\frac{m}{2}\chi^\dagger\sigma^2\chi^*(y) - \cdots\right)\right]$$

$$= i\vec{\sigma}\cdot\vec{\nabla}\chi(\vec{x}) + i\frac{m}{2}\sigma^2\chi^*(\vec{x})\cdot 2$$

$$i\underbrace{(\underset{\sim}{1}\frac{\partial}{\partial t} - \underset{\sim}{\vec{\sigma}}\cdot\vec{\nabla})}_{\bar{\sigma}\cdot\partial}\chi - i m\sigma^2\chi^* = 0$$

$$i\bar{\sigma}\cdot\partial\chi - im\sigma^2\chi^* = 0$$

this equation is solved by elements:

$$\chi^{(+)} = \sqrt{p\cdot\sigma}\,\xi\, e^{-ip\cdot x} \qquad \chi^{(-)} = \sqrt{p\cdot\sigma}\,(-i\sigma^2)\xi^*\, e^{ip\cdot x}$$

$$i\bar{\sigma}\cdot\partial\chi^{(+)} = p\cdot\bar{\sigma}\sqrt{p\cdot\sigma}\,\xi\, e^{-ip\cdot x} = m\sqrt{p\cdot\bar{\sigma}}\,\xi\, e^{-ip\cdot x}$$

$$im\sigma^2\chi^{(-)*} = im\sigma^2(\sqrt{p\cdot\sigma})^*(-i\sigma^2)\,\xi\, e^{-ip\cdot x} = m\sqrt{p\cdot\bar{\sigma}}\,\xi\, e^{-ip\cdot x}$$

so $$i\bar{\sigma}\cdot\partial\chi^{(+)} = im\sigma^2\chi^{(-)*}$$

so write

$$\chi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(\sqrt{p\cdot\sigma}\, \xi^s a_p^s + \sqrt{\hat{p}\cdot\sigma}(-i\sigma^2)\xi^{s*} a_{\hat{p}}^{s\dagger} \right) e^{i\vec{p}\cdot\vec{x}}$$

$$\chi^\dagger(\vec{y}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(\xi^{s\dagger} \sqrt{p\cdot\sigma}\; a_p^{s\dagger} + \xi^{sT} i\sigma^2 \sqrt{\hat{p}\sigma}\; a_{\hat{p}}^s \right) e^{-i\vec{p}\cdot\vec{y}}$$

with $\{ a_p^s , a_q^{s'\dagger} \} = (2\pi)^3 \delta(p-q)\, \delta^{ss'}$

$$\{a_p^s, a_q^{s'}\} = 0 = \{a_p^{\dagger s}, a_q^{\dagger s'}\}$$

$$\{\chi(\vec{x}), \chi^\dagger(y)\} = \int \frac{d^3p\, d^3k}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_k}} \sum_s \Big\{ \sqrt{p\cdot\sigma}\, \xi^s \xi^{\dagger s} \sqrt{k\cdot\sigma}\; (2\pi)^3 \delta(\vec{p}-\vec{k})\, e^{i\vec{p}(\vec{x}-\vec{y})}$$

$$+ \sqrt{\hat{p}\sigma}\, -i\sigma^2 \xi^{s*} \xi^{sT} i\sigma^2 \sqrt{\hat{k}\sigma}\, (2\pi)^3 \delta(p-k)\, e^{ip(\vec{x}-\vec{y})} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{(p\cdot\sigma + \hat{p}\cdot\sigma)}_{2E_p} e^{i\vec{p}\cdot\vec{x}-\vec{y}}$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} = \delta(\vec{x}-\vec{y})$$

$$\{\chi(\vec{x}), \chi(\vec{y})\} = \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_k}} \sum_s \Big\{ \sqrt{p\sigma}\, \xi^s \xi^{s\dagger} (i\sigma^2)(\sqrt{\hat{k}\sigma})^T (2\pi)^3 \delta(\vec{p}+\vec{k})\, e^{ip(x-y)}$$

$$+ \sqrt{\hat{p}\sigma}(-i\sigma^2)\xi^{s*}\xi^T (\sqrt{\hat{k}\sigma})^T 2\pi\delta(\vec{p}+\vec{k})\, e^{ip(\vec{x}-\vec{y})} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big(\underbrace{\sqrt{p\sigma}\sqrt{p\cdot\bar{\sigma}}}_{m} (i\sigma^2) + \underbrace{\sqrt{\hat{p}\sigma}\sqrt{\hat{p}\bar{\sigma}}}_{m} (-i\sigma^2) \Big) e^{ip(\vec{x}-\vec{y})}$$

$$= 0$$

so this formula does represent the anti-commutation relations

Now plug these formulae into H:

$$H = \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_k}} \; e^{-ip\cdot\vec{x}} e^{i\vec{k}\cdot\vec{x}}$$

$\hat{p} = (p^0, -\vec{p})$
$\hat{k} = (k^0, -\vec{k})$

$$\Big\{ \left(\xi^{\dagger s}\sqrt{p\cdot\sigma}\, a_p^{s\dagger} + \xi^{sT} i\sigma^2 \sqrt{\hat{p}\cdot\sigma}\, a_{\hat{p}}^s\right)\left(-\vec{\sigma}\cdot\vec{k}\right)$$
$$\left(\sqrt{k\cdot\sigma}\, \xi^t a_k^t + \sqrt{\hat{k}\cdot\sigma}\,(-i\sigma^2)\, \xi^{t*} a_{\hat{k}}^{t\dagger}\right)$$

$$-i\frac{m}{2}\left(\xi^{sT}(\sqrt{\hat{p}\cdot\sigma})^T a_{\hat{p}}^s + \xi^{s\dagger} i\sigma^2 (\sqrt{p\cdot\sigma})^T a_p^{\dagger s}\right)\sigma^2$$
$$\left(\sqrt{k\cdot\sigma}\, \xi^t a_k^t + \sqrt{\hat{k}\cdot\sigma}\,(-i\sigma^2)\, \xi^{t*} a_{\hat{k}}^{t\dagger}\right)$$

$$+i\frac{m}{2}\left(\xi^{s\dagger}\sqrt{p\cdot\sigma}\, a_p^{s\dagger} + \xi^{sT} i\sigma^2 \sqrt{\hat{p}\cdot\sigma}\, a_{\hat{p}}^s\right)\sigma^2$$
$$\left((\sqrt{\hat{k}\cdot\sigma})^T \xi^{t*} a_{\hat{k}}^{t\dagger} + (\sqrt{k\cdot\sigma})^T(-i\sigma^2)\, \xi^t a_k^t\right)\Big\}$$

first collect the aa terms: $\int d^3x \; e^{-i\vec{p}\cdot\vec{x}} e^{+i\vec{k}\cdot\vec{x}} = (2\pi)^3 \delta(\vec{p}-\vec{k})$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{sT} i\sigma^2 \sqrt{\hat{p}\cdot\sigma}\, a_{\hat{p}}^s \; (-\vec{\sigma}\cdot\vec{k}) \sqrt{p\cdot\sigma}\, \xi^t a_p^t$$
$$+ \frac{m}{2} \xi^{sT} (\sqrt{\hat{p}\cdot\sigma})^T a_{\hat{p}}^s (i\sigma^2) \sqrt{p\cdot\sigma}\, \xi^t a_p^t$$
$$- \frac{m}{2} \xi^{sT} (i\sigma^2) \sqrt{\hat{p}\cdot\sigma}\, (-i\sigma^2) (\sqrt{p\cdot\sigma})^T (-i\sigma^2)\, \xi^t a_p^t \Big\}$$

$$(\hat{p}\cdot\sigma)^T(-i\sigma^2) = -i\sigma^2 (p\cdot\sigma) \quad , \text{ so}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\,T} (i\sigma^2) \sqrt{p\cdot\bar\sigma} \left(\frac{p\cdot\sigma - p\cdot\bar\sigma}{2}\right) \sqrt{p\cdot\sigma}\, \xi^t$$

$$+ \frac{m}{2} \xi^{s\,T} (-i\sigma^2) \sqrt{p\cdot\sigma} \sqrt{p\cdot\sigma}\, \xi^t$$

$$\boxed{\begin{array}{l} p\cdot\bar\sigma \sqrt{p\cdot\sigma} = m\sqrt{p\cdot\bar\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} = m \end{array}} \qquad - \frac{m}{2} \xi^{s\,T} i\sigma^2 \sqrt{p\cdot\bar\sigma} \sqrt{p\cdot\bar\sigma} (-1)\, \xi^t \Big\} a^s_{\hat p} a^t_p$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\,T} (i\sigma^2) \Big\{ \frac{m}{2} (p\cdot\sigma - p\cdot\bar\sigma) - \frac{m}{2} p\cdot\sigma + \frac{m}{2} p\cdot\bar\sigma \Big\} \xi^t$$

$$\cdot\, a^s_{\hat p} a^t_p$$

$$= 0$$

similarly, the $a^\dagger a^\dagger$ terms are:

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\dagger} \sqrt{p\cdot\sigma} (-\vec\sigma\cdot\vec p) \sqrt{\hat p\cdot\sigma} (-i\sigma^2)\, \xi^{t*} \; a^{s\dagger}_p a^{t\dagger}_{\hat p}$$

$$+ \frac{m}{2} \xi^{s\dagger} i\sigma^2 (\sqrt{p\cdot\sigma})^T (-i\sigma^2) \sqrt{\hat p\cdot\sigma} (-i\sigma^2)\, \xi^{t*} \; a^{s\dagger}_p a^{t\dagger}_{\hat p}$$

$$- \frac{m}{2} \xi^{s\dagger} \sqrt{p\cdot\sigma} (-i\sigma^2) (\sqrt{\hat p\cdot\sigma})^T \xi^{t*} a^{s\dagger}_p a^{t\dagger}_{\hat p} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\dagger} \sqrt{p\cdot\sigma} \left(\frac{p\cdot\sigma - p\cdot\bar\sigma}{2}\right) \sqrt{p\cdot\bar\sigma} (-i\sigma^2)\, \xi^{t*}$$

$$+ \frac{m}{2} \cdot \xi^{s\dagger} \sqrt{p\cdot\bar\sigma} \sqrt{p\cdot\bar\sigma} (-i\sigma^2)\, \xi^{t*}$$

$$- \frac{m}{2} \xi^{s\dagger} \sqrt{p\cdot\sigma} \sqrt{p\cdot\sigma} (-i\sigma^2)\, \xi^{t*} \Big\} a^{s\dagger}_p a^{t\dagger}_{\hat p}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\dagger} \left(\frac{m}{2} (p\cdot\sigma - p\cdot\bar\sigma) + \frac{m}{2} p\cdot\bar\sigma - \frac{m}{2} p\cdot\sigma \right) (-i\sigma^2)\, \xi^{t*} \Big\}$$

$$a^{s\dagger}_p a^{t\dagger}_{\hat p}$$

$$= 0$$

now all that remains is:

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\dagger} \sqrt{p\cdot\sigma} \;(-\vec{\sigma}\cdot\vec{p})\; \sqrt{p\cdot\sigma} \;\xi^t\; a_p^{s\dagger} a_p^t$$

$$+ \; \xi^{s^T} i\sigma^2 \sqrt{\hat{p}\cdot\sigma} \;(-\vec{\sigma}\cdot\vec{p})\; \sqrt{\hat{p}\cdot\sigma} \;(-i\sigma^2)\; \xi^{t*} \; a_{\hat{p}}^s \, a_{\hat{p}}^{t\dagger}$$

$$+ \frac{m}{2} \Big(\xi^{s\dagger} (i\sigma^2) (\sqrt{p\cdot\sigma})^T (-i\sigma^2) \sqrt{p\cdot\sigma} \;\xi^t\; a_p^{s\dagger} a_p^t$$

$$+ \; \xi^{s^T} (\sqrt{\hat{p}\sigma})^T (-i\sigma^2) \sqrt{\hat{p}\cdot\sigma} \,(-i\sigma^2)\, \xi^{t*} \; a_{\hat{p}}^s \, a_{\hat{p}}^{t\dagger} \Big)$$

$$- \frac{m}{2} \Big(\; \xi^{s\dagger} \sqrt{p\sigma} \,(-i\sigma^2)\, (\sqrt{p\cdot\sigma})^T (-i\sigma^2)\, \xi^t \;\; a_p^{s\dagger} a_p^t$$

$$+ \; \xi^{s^T} (i\sigma^2) \sqrt{\hat{p}\cdot\sigma} \;\; (-i\sigma^2) (\sqrt{\hat{p}\cdot\sigma})^T \,\xi^{t*} \;\; a_{\hat{p}}^s \, a_{\hat{p}}^{t\dagger} \;\; \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \Big[\xi^{s\dagger} \sqrt{p\cdot\sigma} \;\; \Big(\frac{p\cdot\sigma - p\cdot\bar{\sigma}}{2}\Big) \sqrt{p\cdot\sigma} \;\; \xi^t$$

$$+ \;\; \frac{m}{2} \;\; \xi^{s\dagger} \sqrt{p\cdot\bar{\sigma}} \sqrt{p\cdot\sigma} \;\; \xi^t \; + \; \frac{m}{2} \xi^{s\dagger} \sqrt{p\sigma} \;\sqrt{p\bar{\sigma}} \;\; \xi^t \Big] \, a_p^{s\dagger} a_p^t$$

$$+ \; \Big[\;\; \xi^{s^T} (\sqrt{p\sigma})^T (\vec{\sigma}\cdot\vec{p})^T (\sqrt{p\sigma})^T \xi^{t*} \; a_{-\vec{p}}^s \, a_{-\vec{p}}^{t\dagger}$$

$$+ \;\; \frac{m}{2} \Big(-\xi^{s^T} (\sqrt{p\bar{\sigma}})^T (\sqrt{p\sigma})^T \xi^{t*} - \;\; \xi^{s^T} (\sqrt{p\sigma})^T (\sqrt{p\cdot\bar{\sigma}})^T \xi^{t*} \Big) a_{-\vec{p}}^s a_{-\vec{p}}^{t\dagger} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \Big[\xi^{s\dagger} \big(p\cdot\sigma \cdot \tfrac{1}{2}(p\cdot\sigma) - \tfrac{m^2}{2}\big) \,\xi^t \; + \; m \cdot m \cdot \; \xi^{s\dagger} \xi^t \Big] \; a_p^{s\dagger} a_p^t$$

$$+ \;\; \Big[\;\; \xi^{t\dagger} \sqrt{p\cdot\sigma} \;\; \vec{\sigma}\cdot\vec{p} \;\; \sqrt{p\cdot\sigma} \;\; \xi^s \; - \;\; m \cdot m \;\; \xi^{t\dagger} \xi^s \Big] \; a_{-\vec{p}}^s \, a_{-\vec{p}}^{t\dagger}$$

now, in the second term, let $\quad a_{-\vec{p}}^s a_{-\vec{p}}^{t\dagger} = - a_{-\vec{p}}^{t\dagger} a_{-\vec{p}} \; + \text{const.}$

drop the const, and let $\; s \leftrightarrow t \quad \vec{p} \leftrightarrow -\vec{p}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\dagger} \left[\tfrac{1}{2}(p\cdot\sigma)^2 - \frac{m^2}{2} + m^2 \right] \xi^t$$

$$+ \xi^{s\dagger} \left[\sqrt{p\cdot\bar\sigma}\, \underbrace{\vec\sigma\cdot\vec p}_{p\cdot\bar\sigma - p\cdot\sigma} \sqrt{p\cdot\bar\sigma} + m^2 \right] \xi^t \Big\} a_p^{s\dagger} a_p^t$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \xi^{s\dagger} \left[\tfrac{1}{2}(p\cdot\sigma)^2 + \frac{m^2}{2} + \tfrac{1}{2}(p\cdot\bar\sigma)^2 + m^2/2 \right] \xi^t \, a_p^{s\dagger} a_p^t$$

$$\tfrac{1}{2}\left((p\cdot\sigma)^2 + (p\cdot\bar\sigma)^2\right) = E_p^2 + |\vec p|^2$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \xi^{s\dagger} \underbrace{\left[E_p^2 + |\vec p|^2 + m^2\right]}_{2E_p \cdot E_p \cdot (\underset{\sim}{1})} \xi^t \, a_p^{s\dagger} a_p^t$$

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \, a_p^{s\dagger} a_p^s + (\text{const})$$

as we might have hoped for!

3.) a) $$\delta\mathcal{L} = \partial_\mu \delta\phi^* \partial^\mu \phi + \partial_\mu \phi^* \partial^\mu \delta\phi$$
$$+ \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \chi^\dagger i\bar{\sigma}\cdot\partial\,\delta\chi$$
$$+ \delta F^* F + F^* \delta F$$

insert into this:

$$\delta\phi = -i\epsilon^T\sigma^2\chi \qquad \delta\phi^* = i\chi^\dagger\sigma^2\epsilon^*$$
$$\delta\chi = \epsilon F + \sigma\cdot\partial\phi\,\sigma^2\epsilon^* \qquad \delta\chi^\dagger = F^*\epsilon^\dagger + \varepsilon^T\sigma^2\partial_\mu\phi^\dagger\sigma^\mu$$
$$\delta F = -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi \qquad \delta F^* = i\,\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon$$

$$\delta\mathcal{L} = \partial_\mu(i\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu(-i\epsilon^T\sigma^2\chi)$$
$$+ [(F^*\varepsilon^\dagger) + \varepsilon^T\sigma^2\partial_\mu\phi^\dagger\sigma^\mu]\, i\bar{\sigma}\cdot\partial\chi$$
$$+ \chi^\dagger i\bar{\sigma}\cdot\partial\,(\varepsilon F + \sigma\cdot\partial\phi\,\sigma^2\varepsilon^*)$$
$$+ (-i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi)F^* + F^*(i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon)$$

terms w. ϕ: (w. integration by parts)

$$\partial_\mu(i\chi^\dagger\sigma^2\epsilon^*)\,\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\,\sigma\cdot\partial\phi\,\sigma^2\varepsilon^*$$
$$= -i\chi^\dagger\sigma^2\varepsilon^*\,\partial^2\phi + i\chi^\dagger i\,\partial^2\phi\,\sigma^2\varepsilon^* = 0$$

terms w. ϕ^*:

$$\partial_\mu\phi^*\partial^\mu(-i\epsilon^T\sigma^2\chi) + \varepsilon^T\sigma^2\partial_\mu\phi^\dagger\sigma^\mu\, i\bar{\sigma}\cdot\partial\chi$$
$$= +i\,\partial^2\phi^*\,\epsilon^T\sigma^2\chi - i\,\varepsilon^T\sigma^2\phi^*\,\partial^2\chi = 0$$ after int. by parts.

terms w. F:

$$\chi^\dagger i\bar\sigma\cdot\partial(\varepsilon F) + i\,\partial_\mu\chi^\dagger\bar\sigma^\mu\varepsilon F$$

$$= \chi^\dagger i\bar\sigma\,\partial F\,\varepsilon \quad - i\,\chi^\dagger\bar\sigma\cdot\partial F\,\varepsilon = 0$$

terms w. F^*:

$$F^*\varepsilon^\dagger i\sigma\cdot\partial\chi + (-i)F^*\varepsilon^\dagger i\,\bar\sigma\cdot\partial\chi = 0$$

so $\quad \delta\mathcal{L} = 0$

b) $$\delta(\Delta\mathcal{L}) = m\,\delta\phi\,F + \tfrac{i}{2}m\,\delta\chi^T\sigma^2\chi + m\,\phi\,\delta F + \tfrac{i}{2}m\,\chi^T\sigma^2\delta\chi$$

$$= m\,\delta\phi\,F + im\,\chi^T\sigma^2\delta\chi + m\,\phi\,\delta F$$

$$= m(-i\epsilon^T\sigma^2\chi)F + im\,\chi^T\sigma^2(\varepsilon F + \sigma\cdot\partial\phi\,\sigma^2\varepsilon^*) + m\,\phi\,(-i\epsilon^\dagger\bar\sigma\cdot\partial\chi)$$

$$= -im\,\epsilon^T\sigma^2\chi\,F + im\,\chi^T\sigma^2\varepsilon\,F + im\,\chi^T\sigma^2\,\sigma\cdot\partial\phi\,\sigma^2\varepsilon^* - im\,\phi\,\varepsilon^\dagger\bar\sigma\cdot\partial\chi$$

$$= -im\,\epsilon^T\sigma^2\chi\,F + im\,\varepsilon^T\sigma^2\chi\,F + im(-\partial_\mu\chi^T(\bar\sigma^\mu)^T\varepsilon^*\,\phi - im\,\phi\,\varepsilon^\dagger\bar\sigma\cdot\partial\chi$$

$$= 0 + im\,\varepsilon^\dagger\bar\sigma\cdot\partial\chi\,\phi - im\,\phi\,\varepsilon^\dagger\bar\sigma\cdot\partial\chi$$

$$= 0$$

the Full Lagrangian is

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*F$$
$$+ m\phi F + \tfrac{i}{2}m\,\chi^T\sigma^2\chi + m\phi^* F^* - \tfrac{i}{2}m\,\chi^\dagger\sigma^2\chi^*$$

$$\delta\mathcal{L} = \delta F^*(F + m\phi^*) + \delta F(m\phi + F^*)$$

so $\quad F = -m\phi^* \quad F^* = -m\phi$

using these equations for F, F^*

$$\mathcal{L} = \underbrace{\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi}_{\text{complex KG boson w. mass } m} + \underbrace{\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \tfrac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)}_{\text{Majorana fermion w. mass } m.}$$

c) $$\Delta\mathcal{L} = F_i\frac{\partial W}{\partial\phi_i} + \frac{i}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\chi_j$$

$$\delta(\Delta\mathcal{L}) = F_i\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}(-i\epsilon^T\sigma^2\chi_j) + \frac{\partial W}{\partial\phi_i}(-i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi_i)$$
$$+ i\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\chi_j^T\sigma^2(\epsilon F_i + \sigma\cdot\partial\phi_i\,\sigma^2\epsilon^*)$$
$$+ \frac{i}{2}\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}\chi_i^T\sigma^2\chi_j\,(-i\epsilon^T\sigma^2\chi_k)$$

the terms w. F are:

$$(-i\,\epsilon^T\sigma^2\chi_j)\,F_i\,\frac{\partial^2 W}{\partial\phi_i\partial\phi_j} + i\,\chi_j^T\sigma^2\epsilon\,F_i\,\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}$$

$$= 0$$

the terms w/ $1\ \chi$ are:

$$-i\,\frac{\partial W}{\partial\phi_i}\,\epsilon^\dagger\bar\sigma\cdot\partial\chi_i + i\,\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\,\chi_j^T\sigma^2\,\sigma\cdot\partial\phi_i\,\sigma^2\epsilon^*$$

$$= -i\,\frac{\partial W}{\partial\phi_i}\,\epsilon^\dagger\bar\sigma\cdot\partial\chi_i + i\,\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\,\underbrace{\chi_j^T\,(\bar\sigma)^T\partial\phi_i\,\epsilon^*}_{=\,-\epsilon^\dagger\bar\sigma^\mu\chi_j\,\cdot\,\partial_\mu\phi_i}$$

$$= -i\,\epsilon^\dagger\bar\sigma\,\partial^\mu\chi_i\,\frac{\partial W}{\partial\phi_i} - i\,\epsilon^\dagger\bar\sigma^\mu\chi_i\,\partial_\mu\left(\frac{\partial W}{\partial\phi_i}\right)$$

$= 0$ after ~~integration~~ by parts.

finally, the last term involves:

$$\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}\,\chi_{i\alpha}\,\chi_{j\beta}\,\chi_{k\gamma}$$

since the derivative is totally symmetric under any interchange of (ijk), but χ anticommute, this object is totally antisymmetric in $[\alpha\beta\gamma]$. But $\alpha,\beta,\gamma = 1,2$ only. So this object must be 0.

$$\delta(\Delta\mathcal{L}) = 0 \quad \text{and so} \quad \delta\mathcal{L} = 0$$

for the special case in the problem set

$$\mathcal{L} = \partial_\mu\phi^\dagger \partial^\mu \phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^\dagger F$$

$$+ F g\phi^2 + ig\phi\, \chi^T \sigma^2 \chi + h.c.$$

then $\quad F^* + g\phi^2 = 0 \qquad F^* = -g\phi^2$

$$\mathcal{L} = \partial_\mu\phi^\dagger \partial^\mu \phi - g^2 |\phi^*\phi|^2 + \chi^\dagger i\bar{\sigma}\cdot\partial\chi$$

$$+ ig\phi\, \chi^T \sigma^2 \chi - ig\,\phi^*\, \chi^\dagger \sigma^2 \chi^*$$

This is a nonlinear field thy with ϕ^4 and Yukawa interactions, both given by the dimensionless coupl g.

$$-\partial^2\phi - ig\,\chi^\dagger \sigma^2 \chi^* - 2g^2 \phi^* \phi^2 = 0$$

$$i\bar{\sigma}\cdot\partial\chi - 2i\,\sigma^2 \chi^* = 0$$

are the equs. of motion.