Physics 330 - Problem Set #2

Solutions

1.) a.) $$\delta S = \int d^4x \, \{ \partial_\mu \delta\phi^* \partial^\mu \phi + \partial_\mu \phi^* \partial^\mu \delta\phi - m^2 \delta\phi^* \phi - m^2 \phi^* \delta\phi \}$$

$$= \int d^4x \, [\delta\phi^* \{ -\partial_\mu \partial^\mu \phi - m^2 \phi \} + \{ \partial^\mu \partial_\mu \phi^* - m^2 \phi^* \} \delta\phi]$$

so

$\delta S = 0 \Rightarrow$ coeff's of $\delta\phi^*$, $\delta\phi = 0$

$$\Rightarrow (\partial_\mu \partial^\mu + m^2)\phi = 0 = (\partial^\mu \partial_\mu + m^2)\phi^*$$

b.) c.) Consistently with the discussion in class, it suffices to show that, using the given commutation relations and Hamiltonian, ϕ and ϕ^* satisfy the Klein-Gordon eq. as their Heisenberg equations of motion.

So, compute

$$i\frac{\partial}{\partial t}\phi = [\phi(x), H]$$

$$= \int d^3y \; [\phi(x), \pi^*(y)\pi(y)]$$

$$= \int d^3y \; i\delta(\vec{x}-\vec{y}) \; \pi(y) = i\pi(x)$$

$$i\frac{\partial}{\partial t}\phi^* = [\phi^*(x), H]$$

$$= \int d^3y \; [\phi^*(x), \pi^*(y)\pi(y)]$$

$$= \int d^3y \; \pi^*(y) \; [\phi^*(x), \pi(y)]$$

$$= \int d^3y \; \pi^*(y) \, i\delta(\vec{x}-\vec{y}) = i\pi^*(x)$$

$$i\frac{\partial}{\partial t}\pi(x) = [\pi(x), H]$$

$$= [\pi(x), \int d^3y \; (\vec{\nabla}\phi^* \vec{\nabla}\phi + m^2\phi^*\phi)]$$

integrate by parts $= [\pi(x), \int d^3y \; \phi^*(y) \, (-\nabla^2\phi + m^2\phi)(y)$

$$= (-i)(-\nabla^2\phi + m^2\phi)(x)$$

$$i\frac{\partial}{\partial t}\pi^*(x) = [\pi^*(x), H]$$

$$= [\pi^*(x), \int d^3y \{-\nabla^2\phi^* \; \phi + m^2\phi^*\phi\}]$$

$$= (-i)\,(-\nabla^2\phi^* + m^2\phi^*)(x)$$

$$\frac{\partial}{\partial t}\phi = \pi \qquad \frac{\partial}{\partial t}\pi = \nabla^2\phi - m^2\phi$$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2}\phi = \nabla^2\phi - m^2\phi$$

$$\text{or} \quad \partial^\mu\partial_\mu\phi + m^2\phi = 0$$

similarly for ϕ^*

$$\frac{\partial}{\partial t}\phi^* = \pi^* \qquad \frac{\partial}{\partial t}\pi^* = \nabla^2\phi^* - m^2\phi^*$$

$$\Rightarrow \quad (\partial_\mu\partial^\mu + m^2)\phi^* = 0$$

d.) Making a slight generalization of what we did in class, write

$$\phi(\vec{x}) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}} + b^\dagger_{-\vec{p}}\right)e^{+i\vec{p}\cdot\vec{x}}$$

$$\phi^*(\vec{x}) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(b_{\vec{p}} + a^\dagger_{-\vec{p}}\right)e^{i\vec{p}\cdot\vec{x}}$$

$$\pi(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(-iE_p)\left(a_{\vec{p}} - b^\dagger_{-\vec{p}}\right)e^{i\vec{p}\cdot\vec{x}}$$

$$\pi^*(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(-iE_p)\left(b_{\vec{p}} - a^\dagger_{-\vec{p}}\right)e^{i\vec{p}\cdot\vec{x}}$$

with $[a_p, a^\dagger_q] = (2\pi)^3\delta(p-q) = [b_p, b^\dagger_q]$

and $a, a^\dagger$ <u>commuting</u> with $b, b^\dagger$

then manifestly $[\phi(x), \pi(y)] = [\phi^*(x), \pi^*(y)] = 0$

$$[\phi(x), \phi^*(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} e^{i\vec{p}\cdot x} e^{i\vec{k}\cdot\vec{y}} \frac{1}{\sqrt{2\epsilon_p 2E_k}}$$

$$\cdot \quad [(a_p + b^+_{-p}), (b_{\vec{k}} + a^+_{-\vec{k}})]$$

the second line is

$$(2\pi)^3 \delta(\vec{p}+\vec{k}) - (2\pi)^3 \delta(\vec{p}+\vec{k}) = 0$$

similarly: $[\pi(x), \pi^*(y)] \propto [(a_p - b^+_{-\vec{p}}), (b_{\vec{k}} - a^+_{-\vec{k}})] = 0$

$$[\phi(x), \pi^*(y)] = \int \frac{d^3p\, d^3k}{(2\pi)^3 (2\pi)^3} \frac{1}{\sqrt{2\epsilon_q 2\epsilon_k}} e^{i\vec{p}\cdot\vec{x}} e^{i\vec{k}\cdot\vec{y}} (-iE_k)$$

$$[a_{\vec{p}} + b^+_{-\vec{p}}, b_{\vec{k}} - a^+_{-\vec{k}}]$$

$$= \int \frac{d^3p\, d^3k}{(2\pi)^6} \frac{1}{\sqrt{2\epsilon_p 2E_k}} e^{ipx} e^{iky} (-iE_k) (2\pi)^3 \delta(\vec{p}+\vec{k}) \cdot (-2)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{i\, \cancel{2}E_p}{\cancel{2}E_p} e^{i\vec{p}(\vec{x}-\vec{y})} = i\, \delta(\vec{x}-\vec{y})$$

and

$[\phi^*(x), \pi(y)] = i\,\delta(x-y)$ similarly

$$H = \int d^3x \int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_p 2E_k}}\; e^{i(\vec{p}+\vec{k})\cdot\vec{x}}$$

$$\Big\{ (-iE_p)(-iE_k)\,(b_{\vec{p}} - a^{+}_{-\vec{p}})(a_{k} - b^{+}_{-\vec{k}})$$

$$+ (i\vec{p}\,)(i\vec{k}\,)\;(b_{\vec{p}} + a^{+}_{-\vec{p}})(a_k + b^{+}_{-\vec{k}})$$

$$+ \quad m^2 \qquad (b_{\vec{p}} + a^{+}_{-\vec{p}})(a_k + b^{+}_{-\vec{k}}) \Big\}$$

now, the integral over d^3x gives $(2\pi)^3\delta(\vec{p}+\vec{k})$, so set

$\vec{p} = -\vec{k}$

$$= \int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\Big\{ -E_k^2\big\{-b_{-\vec{k}}b^{+}_{-\vec{k}} - a^{+}_k a_k + b_{-\vec{k}}a_k + a^{+}_{\vec{k}}b^{+}_{-\vec{k}}\big\}$$

$$+ (k^2+m^2)\big\{ b_{-k}b^{+}_{-k} + a^{+}_k a_k \pm b_{-\vec{k}}a_k + a^{+}_{\vec{k}}b^{+}_{-\vec{k}} \big\} \Big\}$$

the ba $a^{+}b^{+}$ terms cancel, since $E_k^2 = (k^2+m^2)$. Then

$$= \int\frac{d^3k}{(2\pi)^3} E_k\,(b_k b_k^{+} + a_k^{+}a_k)$$

$$H = \int\frac{d^3k}{(2\pi)^3} E_k\,(b_k^{+}b_k + a_k^{+}a_k) \quad + \quad (\text{const.})$$

Let $|0\rangle$ be st. $a_k|0\rangle = b_k|0\rangle = 0$; then

there are two types of single-particle states:

$$|b(\vec{k})\rangle = \sqrt{2E_k}\, a_k^\dagger|0\rangle \qquad |\bar{b}(\vec{k})\rangle = \sqrt{2E_k}\, b_k^\dagger|0\rangle$$

these states both have $E = E_k = [k^2+m^2]^{1/2}$. Since $\phi(x)$ <u>destroys</u> $|b(\vec{k})\rangle$ and <u>creates</u> $|\bar{b}(k)\rangle$, these are particle and antiparticle.

e.)
$$Q = -i\int d^3x \int \frac{d^3p\, d^3k}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_k}}\, e^{i\vec{p}\cdot\vec{x}} e^{i\vec{k}\cdot\vec{x}}$$

$$\{ (-iE_p)(b_{\vec{p}} - a^\dagger_{-\vec{p}})(a_{\vec{k}} + b^\dagger_{-\vec{k}})$$
$$- (-iE_k)(b_p + a^\dagger_{-p})(a_k - b^\dagger_{-k}) \}$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\{ -E_p (b_{\vec{p}} b^\dagger_{\vec{p}} - a^\dagger_{-\vec{p}} a_{-\vec{p}} + b_{\vec{p}} a_{-\vec{p}} - a^\dagger_{-\vec{p}} b^\dagger_{p})$$
$$+ E_p(-b_p b_p^\dagger + a^\dagger_{-p} a_p + b_p a_{-p} - a^\dagger_{-p} b^\dagger_p)\}$$

again ba $a^\dagger b^\dagger$ terms cancel.

$$= \int\frac{d^3p}{(2\pi)^3}\frac{2E_p}{2E_p}(a_p^\dagger a_p - b_p b_p^\dagger)$$

up to a constant

$$Q = \int\frac{d^3p}{(2\pi)^3}(a_p^\dagger a_p - b_p^\dagger b_p)$$

and obviously

$$[Q, H] = 0$$

so

$$Q\, a_p^\dagger |0\rangle = +1 \cdot a_p^\dagger |0\rangle$$

$$Q\, b_p^\dagger |0\rangle = -1 \cdot b_p^\dagger |0\rangle$$

Q is a charge : particles have charge +1, antiparticles have charge −1.

2.) a.) Generalizing the above:

$$[\phi_a(\vec{x}), \pi_b^*(\vec{y})] = i\,\delta(\vec{x}-\vec{y})\,\delta_{ab} = [\phi_a^*(\vec{x}), \pi_b(\vec{y})]$$

$$[\phi_a(\vec{x}), \phi_b^{(*)}(\vec{y})] = 0 = [\pi_a(\vec{x}), \pi_b^{(*)}(\vec{y})]$$

$$H = \int d^3y \left\{ \pi_a^* \pi_a + \vec{\nabla}\phi_a^* \vec{\nabla}\phi_a + m^2 \phi_a^* \phi_a \right\}$$

sum over a understood.

since ϕ_a π_a^* etc. commute for different values of a, the above construction goes through separately for $a=1,2$.

thus, with

$$[a_{pa}, a^\dagger_{qb}] = (2\pi)^3 \delta(\vec{p}-\vec{q})\, \delta_{ab} = [b_{pa}, b^\dagger_{qb}]$$

$$\phi_a(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{pa} + b^\dagger_{-\vec{p}a} \right) e^{i\vec{p}\cdot\vec{x}}$$

etc

diagonalizes H to the form

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \left(a^\dagger_{pa} a_{pa} + b^\dagger_{pa} b_{pa} \right) + \text{const.}$$

the time-dependent field $\phi(x)$ is

$$\phi_a(x) = e^{iHx^0} \phi_a(\vec{x})\, e^{-iHx^0}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{pa}\, e^{-ip\cdot x} + b^\dagger_{pa}\, e^{ip\cdot x} \right)$$

so $\phi_a(x)$ satisfies the Klein Gordon eq.

b.) Follow the argument on p. 6

$$Q = -i \int d^3x \int \frac{d^3p\, d^3k}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_k}} e^{i\vec{p}\cdot\vec{x}} e^{i\vec{k}\cdot\vec{x}}$$

$$\cdot \left\{ -i E_p \left(b_{pa} - a^{\dagger}_{-pa}\right)\left(a_{ka} + b^{\dagger}_{-ka}\right) + i E_k \left(b_{pa} + a^{\dagger}_{-pa}\right)\left(a_{ka} - b^{\dagger}_{-ka}\right) \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ -E_p \left(b_{pa} b^{\dagger}_{pa} - a^{\dagger}_{-pa} a_{-pa} + b_{pa} a_{-pa} - a^{\dagger}_{-pa} b^{\dagger}_{pa}\right) + E_p \left(-b_{pa} b^{\dagger}_{pa} + a^{\dagger}_{-pa} a_{-pa} + b_{pa} a_{-pa} - a^{\dagger}_{-pa} b^{\dagger}_{pa}\right) \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(b^{\dagger}_{pa} b_{pa} - a^{\dagger}_{pa} a_{pa} \right) + (\text{const})$$

this manifestly commutes with $\mathcal{H}$ on p. 8.

c.) we can compute Q^j by the same method.

$$Q^j = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ -E_p \left(b_{pa} \text{[redacted]} a^{\dagger}_{-pa}\right) \frac{\sigma^j_{ab}}{2} \left(a_{pb} + b^{\dagger}_{pb}\right) + E_p \left(b_{pa} + a^{\dagger}_{-pa}\right) \frac{\sigma^j_{ab}}{2} \left(a_{-pb} - b^{\dagger}_{pb}\right) \right\}$$

again the $a^{\dagger}b^{\dagger}$, ba terms cancel.

$$Q^j = \int \frac{d^3p}{(2\pi)^3} \left(a^\dagger_{pa} \frac{\sigma^j_{ab}}{2} a_{pb} - b_{pa} \frac{\sigma^j_{ab}}{2} b^\dagger_{pb} \right)$$

Let me show carefully that this commutes with H

$$[H, Q^j] = \left[\int \frac{d^3k}{(2\pi)^3} E_k (a^\dagger_{kc} a_{kc} + b^\dagger_{kc} b_{kc}) , Q^j \right]$$

$$= \int \frac{d^3k}{(2\pi)^3} E_k \int \frac{d^3p}{(2\pi)^3} \Big\{ \left[a^\dagger_{kc} a_{kc} , a^\dagger_{pa} \frac{\sigma^j_{ab}}{2} a_{pb} \right] - \left[b^\dagger_{kc} b_{kc} , b_{pa} \frac{\sigma^j_{ab}}{2} b^\dagger_{pb} \right] \Big\}$$

$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} E_k \Big\{ a^\dagger_{kc} \left[(2\pi)^3 \delta(\vec{k}-\vec{p}) \delta_{ca} \right] \frac{\sigma^j_{ab}}{2} a_{pb}$$
$$+ a^\dagger_{pa} \frac{\sigma^j_{ab}}{2} \left[-(2\pi)^3 \delta(\vec{k}-\vec{p}) \delta_{cb} \right] a_{kc}$$
$$- b^\dagger_{kc} b_{pa} \frac{\sigma^j_{ab}}{2} \left[(2\pi)^3 \delta(\vec{k}-\vec{p}) \delta_{bc} \right]$$
$$- \left(-(2\pi)^3 \delta(k-p) \delta_{ca} \right) \frac{\sigma^j_{ab}}{2} b^\dagger_{pb} b_{kc} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \Big\{ a^\dagger_{pa} \frac{\sigma^j_{ab}}{2} a_{pb} - a^\dagger_{pa} \frac{\sigma^j_{ab}}{2} a_{pb} - b^\dagger_{pb} b_{pa} \frac{\sigma^j_{ab}}{2} + b^\dagger_{pb} b_{pa} \frac{\sigma^j_{ab}}{2} \Big\}$$

$$= 0$$

$$[Q^j, Q^k] = \int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3} \Big\{ \Big[a^+_{pa}\frac{\sigma^j_{ab}}{2}a_{pb},\ a^+_{kc}\frac{\sigma^k_{cd}}{2}a_{kd}\Big]$$

$$+ \Big[b_{pa}\frac{\sigma^j_{ab}}{2}b^+_{pb},\ b_{kc}\frac{\sigma^k_{cd}}{2}b^+_{kd}\Big]\Big\}$$

$$= \int \frac{d^3p\, d^3k}{(2\pi)^6} \Big\{ a^+_{pa}\ \frac{\sigma^j}{2}_{ab}\left((2\pi)^3\delta(\vec p-\vec k)\delta_{bc}\right)\frac{\sigma^k}{2}_{cd}\ a_{kd}$$

$$+ a^+_{kc}\ \frac{\sigma^k_{cd}}{2}\left(-(2\pi)^3\delta(\vec p-\vec k)\delta_{ad}\right)\frac{\sigma^j}{2}_{ab}\ a_{pb}$$

$$+ b_{pa}\ \frac{\sigma^j_{ab}}{2}\left(-(2\pi)^3\delta(\vec p-\vec k)\delta_{bc}\right)\frac{\sigma^k_{cd}}{2}\ b^+_{kd}$$

$$+ b_{kc}\ \frac{\sigma^k_{cd}}{2}\left((2\pi)^3\delta(\vec p-\vec k)\delta_{ad}\right)\frac{\sigma^j_{ab}}{2}\ b^+_{pb}\Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3}\Big\{ a^+_{pa}\left[\frac{\sigma^j}{2}\frac{\sigma^k}{2} - \frac{\sigma^k}{2}\frac{\sigma^j}{2}\right]_{ab} a_{pb}$$

$$+ b_{pa}\left[-\frac{\sigma^j}{2}\frac{\sigma^k}{2} + \frac{\sigma^k}{2}\frac{\sigma^j}{2}\right]_{ab} b^+_{pb}\Big\}$$

now $\quad [\sigma^j, \sigma^k] = 2i\,\epsilon^{jkl}\sigma^l$

a $\quad \left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}\right] = i\,\epsilon^{jkl}\frac{\sigma^l}{2} \quad$ so

$$= \int \frac{d^3p}{(2\pi)^3}\ i\epsilon^{jkl}\Big\{ a^+_{pa}\ \frac{\sigma^l}{2}_{ab}\ a_{pb} - b_{pa}\frac{\sigma^l_{ab}}{2} b^+_{pb}\Big\}$$

so

$$[Q^j, Q^k] = i\,\epsilon^{jkl}\,Q^l$$

this is the algebra of SU(2) or the rotation group.

3.) a.)

$$L^1 = \tfrac{1}{2}(J^{23} - J^{32}) = J^{23}$$

$$L^2 = J^{31} = -J^{13}$$

$$L^3 = J^{12} = -J^{21}$$

so compute:

$$[L^1, L^2] = [J^{23}, J^{31}] = i(\overset{-1}{g^{33}} J^{21} - \overset{0}{g^{23}} J^{31} - \overset{0}{g^{31}} J^{23} + \underset{0}{g^{21}} J^{33})$$

$$= -i J^{21} = +i L^3$$

$$[L^2, L^3] = [J^{31}, J^{12}] = i(g^{11} J^{32} + \text{zero})$$

$$= i L^1$$

similarly $[L^i, L^j] = i\,\epsilon^{ijk}\,L^k$

$$[K^1, K^2] = [J^{01}, J^{02}] = i[g^{10} J^{02} - g^{00} J^{12} - g^{12} J^{00} + g^{02} J^{10}]$$

$$= -i\,g^{00} J^{12} = -i L^3$$

similarly: $[K^i, K^j] = -i\,\epsilon^{ijk} L^k$

$$[K^1, L^2] = [J^{01}, J^{31}]$$

$$= i\,(g^{13} J^{01} - g^{03} J^{11} - g^{11} J^{03} + g^{01} J^{13})$$

$$= +i\, J^{03} = i\, K^3$$

similarly $[K^i, L^j] = [L^i, K^j] = i\,\epsilon^{ijk} K^k$

in all

$$[L^i, L^j] = i\,\epsilon^{ijk} L^k$$
$$[L^i, K^j] = [K^i, L^j] = i\,\epsilon^{ijk} K^k$$
$$[K^i, K^j] = -i\,\epsilon^{ijk} L^k$$

then if $J_+^i = \frac{1}{2}(L^i + i K^i)$

$$[J_+^i, J_+^j] = \frac{1}{4}[(L^i + iK^i), (L^j + i K^j)]$$

$$= \frac{1}{4}\, i\,\epsilon^{ijk} \{ L^k + 2i\, K^k + (i)^2 (-L)^k \}$$

$$= i\,\epsilon^{ijk}\, \frac{1}{2}(L^k + i K^k) = i\,\epsilon^{ijk} J_+^k$$

similarly if $J_-^i = \frac{1}{2}(L^i - i K^i)$

$$[J_-^i, J_-^j] = i\epsilon^{ijk} J_-^k$$

finally
$$[J_+^i, J_-^j] = \tfrac{1}{4}[L^i + iK^i, L^j - iK^j]$$
$$= i\epsilon^{ijk}\tfrac{1}{4}[L^k + iK^k - iK^k + (-L^k)]$$
$$= 0$$

so J_+^i J_-^i commute and separately satisfy the CR's of angular momentum.

b.) In the $(\frac{1}{2},0)$ representation of (J_+, J_-)

$$J_+^i = \frac{\sigma^i}{2} \qquad J_-^i = 0$$

then
$$L^i = J_+^i + J_-^i = \frac{\sigma^i}{2}$$
$$K^i = -i(J_+^i - J_-) = -i\,\frac{\sigma^i}{2}$$

the Lorentz transformation in this representation is

$$(1 - i\vec{\theta}\cdot\vec{L} - i\vec{\beta}\cdot\vec{K}) = \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)$$

ie. $\underset{\text{(2-compont)}}{\psi} \to (1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2})\psi$

this is the ψ_L transformat.

In the $(0\frac{1}{2})$ representatn

$$J_+^i = 0 \qquad J_-^i = \frac{\sigma^i}{2}$$

$$J^i = \frac{\sigma^i}{2} \qquad K^i = +i\frac{\sigma^i}{2}$$

$$\psi \to (1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\vec{\sigma}/2)\psi$$

this is the ψ_R transformatn.

c.) In the $(\frac{1}{2},\frac{1}{2})$ representatn, we have a 2×2 matrix V that transforms under Lorentz transformations as

$$V \to (1 - i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\beta}\cdot\vec{\sigma}/2)\, V\, (1 + i\vec{\theta}\cdot\vec{\sigma}/2 + \beta\cdot\vec{\sigma}/2)$$

Let's write this more explicitly:

$$V = \begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} = V^0 + \vec{V}\cdot\vec{\sigma}$$

then under rotations:

$$V \to (1 - i\vec{\theta}\cdot\vec{\sigma}/2)\ (V^0 + \vec{V}\cdot\vec{\sigma})\ (1 + i\vec{\theta}\cdot\vec{\sigma}/2)$$

$$= V^0 + \vec{V}\cdot\vec{\sigma} - i\,\theta^i V^j\,[\tfrac{\sigma^i}{2}, \sigma^j]$$

$$= V^0 + \vec{V}\cdot\vec{\sigma} - i\,\theta^i V^j\ i\epsilon^{ijk}\sigma^k$$

$$= V^0 + (\vec{V} + \vec{\theta}\times\vec{V})\cdot\vec{\sigma}$$

under boosts.

$$V \to (1 + \vec{\beta}\cdot\vec{\sigma}/2)(V^0 + \vec{V}\cdot\vec{\sigma})(1 + \vec{\beta}\cdot\vec{\sigma}/2)$$

$$= V^0 + \vec{V}\cdot\vec{\sigma} + V^0\vec{\beta}\cdot\vec{\sigma} + \frac{\beta^i}{2}V^j\underbrace{\{\sigma^i,\sigma^j\}}_{2\delta^{ij}}$$

$$= (V^0 + \vec{\beta}\cdot\vec{V}) + (\vec{V} + \beta V^0)\cdot\vec{\sigma}$$

under rotations

$$V^0 \to V^0 \qquad \vec{V} \to \vec{V} + \vec{\theta}\times\vec{V}$$

under boosts
$$\begin{pmatrix} V^0 \\ \hat{\beta}\cdot\vec{V}\end{pmatrix} \to \begin{pmatrix} 1 & \beta \\ \beta & 1\end{pmatrix}\begin{pmatrix} V^0 \\ \hat{\beta}\cdot\vec{V}\end{pmatrix}$$

this is the transformation law of a 4-vector!