# Physics 330 – Problem Set 1
## Solutions

1.) I set $c = 1$, so

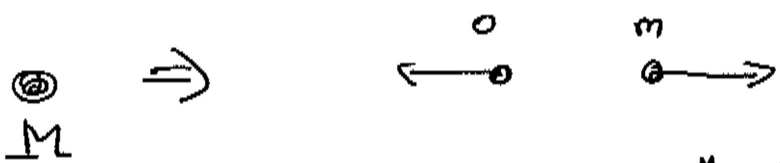
for a massless particle moving $\parallel \hat{3}$:

$$p^\mu = (p, 0, 0, p)$$

for a massive particle moving $\parallel \hat{3}$

$$p^\mu = (E_p, 0\ 0\ p) \qquad E_p = \left[|\vec{p}|^2 + m^2\right]^{1/2}$$

a).




at rest $\qquad p^\mu = (p, 0\,0\, -p) \qquad p^\mu = (E_p, 0, 0, p)$

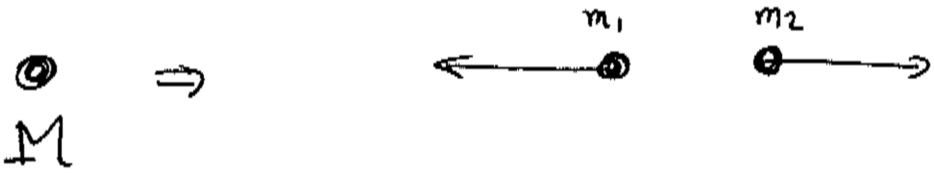
$$p + E_p = M \qquad \text{conserv. of energy.}$$

then
$$[p^2 + m^2]^{1/2} = M - p$$

$$\cancel{p^2} + m^2 = M^2 - 2Mp + \cancel{p^2}$$

$$p = \frac{M^2 - m^2}{2M} \qquad E_p = \frac{M^2 + m^2}{2M}$$

energy + mom of massless particle $\qquad$ energy of massive particle b.)




$$p_1 = (E_1, 0, 0, -p) \qquad p_2 = (E_2, 0, 0, p)$$

$$E_1 = (p^2 + m_1^2)^{1/2} \qquad E_2 = (p^2 + m_2^2)^{1/2}$$

$$E_1 + E_2 = M$$

$$E_1^2 + E_2^2 + 2E_1E_2 = M^2$$

$$4E_1^2E_2^2 = (M^2 - E_1^2 - E_2^2)^2$$

$$4E_1^2E_2^2 = M^4 - 2M^2(E_1^2 + E_2^2) + (E_1^2 + E_2^2)^2$$

$$0 = M^4 - 2M^2(E_1^2 + E_2^2) + (E_1^2 - E_2^2)^2$$

$$0 = M^4 - 2M^2(2p^2 + m_1^2 + m_2^2) + (\cancel{p^2} + m_1^2 - \cancel{p^2} + m_2^2)^2$$

so

$$4p^2M^2 = M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)$$

$$p = \frac{[M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)]^{1/2}}{2M}$$

$$E_1^2 = p^2 + m_1^2 = \frac{M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 + 4M^2m_1^2}{4M^2}$$

$$= \frac{M^4 + 2M^2(m_1^2 - m_2^2) + (m_1^2 - m_2^2)^2}{4M^2} = \left(\frac{M^2 + m_1^2 - m_2^2}{2M}\right)^2$$

so

$$p = \frac{\left[M^4 - 2M^2(m_1^2+m_2^2) + (m_1^2-m_2^2)^2\right]^{\frac{1}{2}}}{2M}$$

$$E_1 = \frac{M^2+m_1^2-m_2^2}{2M} \qquad E_2 = \frac{M^2+m_2^2-m_1^2}{2M}$$

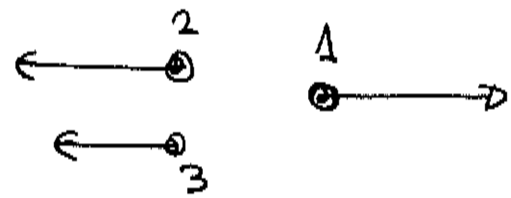
One sometimes sees the notation:

$$\lambda(a,b,c) = a^2+b^2+c^2 - 2ab - 2ac - 2bc$$

$$p = \frac{\sqrt{\lambda(M^2, m_1^2, m_2^2)}}{2M}$$

c.) To answer this question, we should ask: What configurations allow the energies $E_1$ $E_2$ $E_3$ to take their maximum values?

For $E_1$, this configuration is




1 recoils against 2 + 3: $\quad p_1 = p_2 + p_3$

For all massless particles $\quad p_1 = E_1 \quad p_2 = E_2 \quad p_3 = E_3$

$$E_1 + E_2 + E_3 = M \quad \text{so} \quad p_1 = M/2$$

In terms of the $x_i$ $\qquad x_i = 2E_i/M$

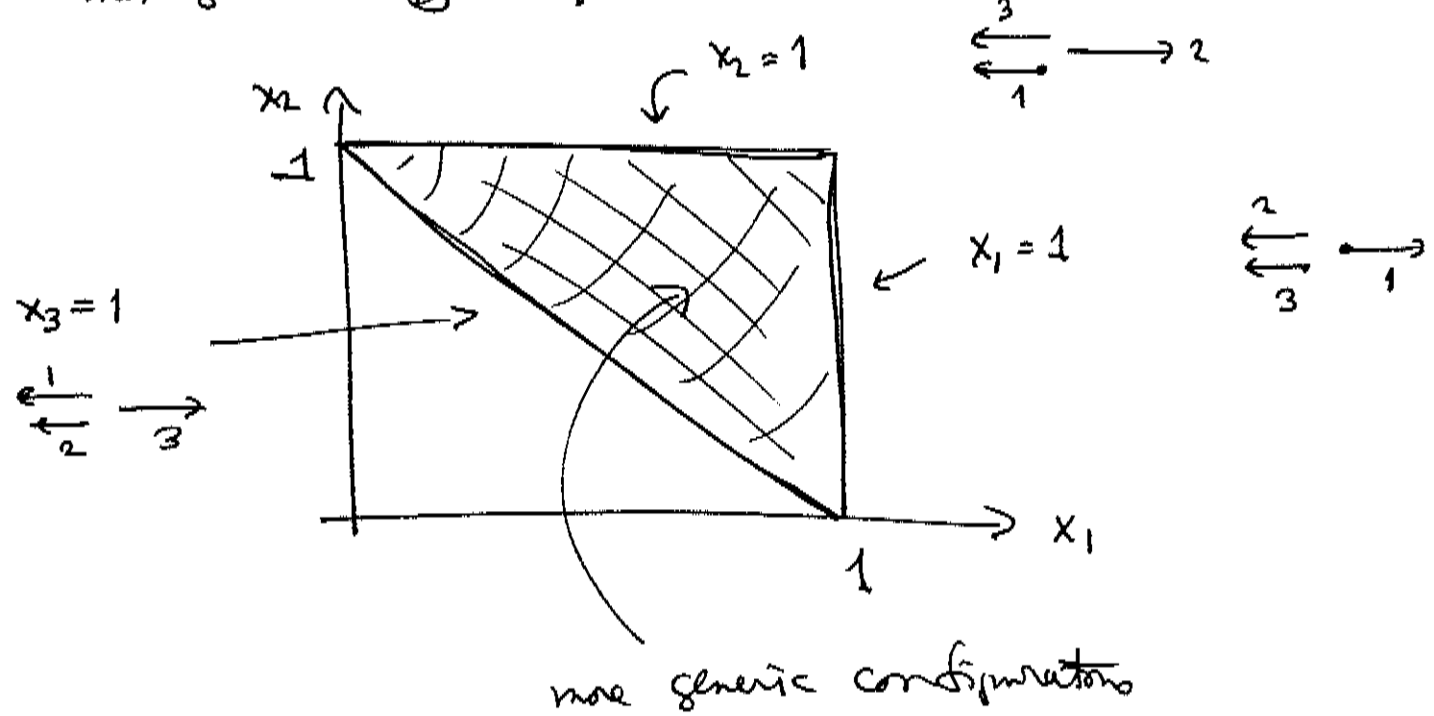
$$x_1 = 1 \qquad x_2 + x_3 = 1$$

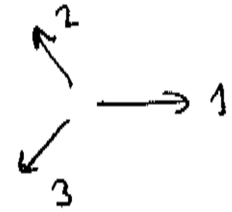
The other boundaries correspond to

$$x_2 = 1 \qquad x_1 + x_3 = 1$$

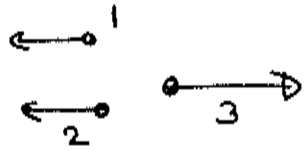
$$x_3 = 1 \qquad x_1 + x_2 = 1$$

This gives a region of the $x_1\, x_2$ plane:



d.) If particle 3 has mass $m$:




now has $\quad p_3 = p_1 + p_2 \qquad E_3 = (p_3^2 + m^2)^{1/2}$

1+2 is a massless 4-vector $\quad (p_1+p_2)^2 = 0$ or

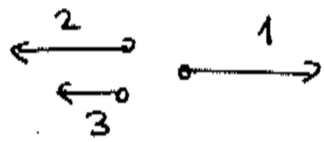
$(p_1+p_2)^\mu = (p_3, 0, 0, -p_3)$. Then this is

the problem we solved in part (a).

$$p_3 = \frac{M^2 - m^2}{2M} \quad \Rightarrow \quad x_1 + x_2 = \left(1 - \frac{m^2}{M^2}\right)$$

$$E_3 = \frac{M^2 + m^2}{2M}$$

For the configuration




$$p_1 = p_2 + p_3$$

$$M = p_1 + p_2 + (p_3^2 + m^2)^{1/2}$$

$$M - p_1 - p_2 = ((p_1 - p_2)^2 + m^2)^{1/2}$$

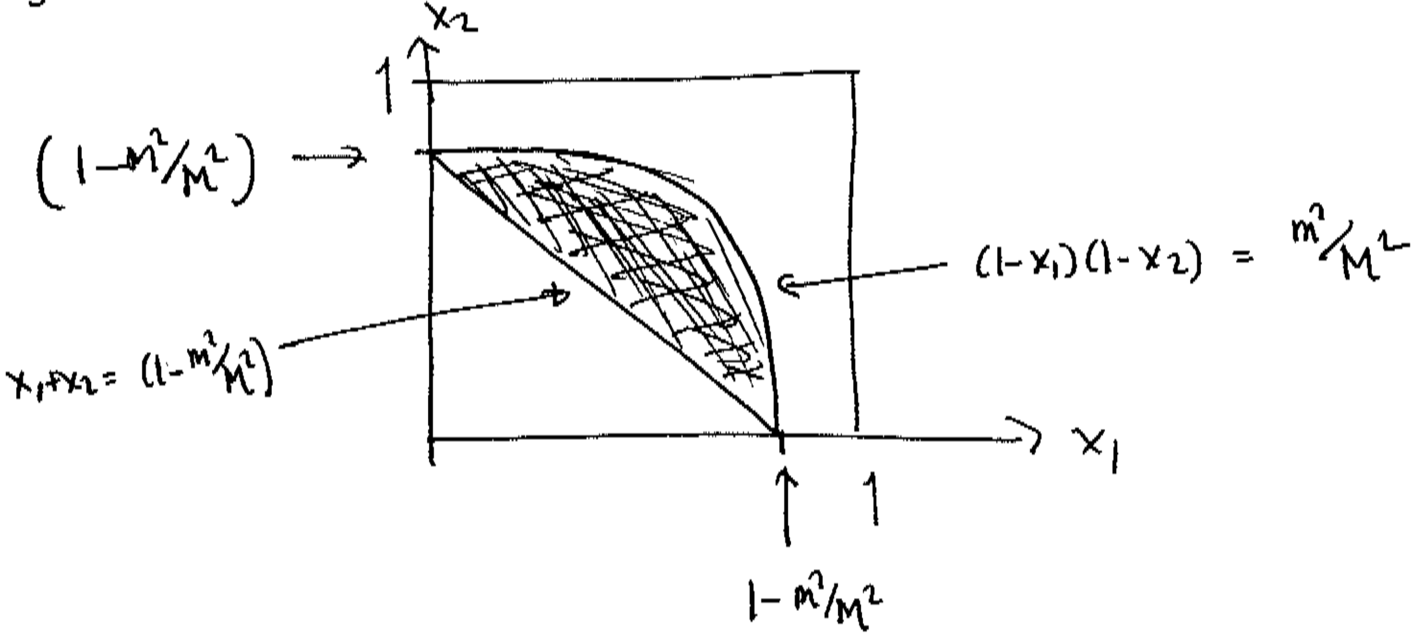
$$M^2 - 2M(p_1+p_2) + (p_1+p_2)^2 = (p_1 - p_2)^2 + m^2$$

$$M^2 - 2M(p_1+p_2) + 4p_1p_2 = m^2$$

$$(M - 2p_1)(M - 2p_2) = m^2$$

$$(1 - x_1)(1 - x_2) = m^2/M^2$$

so the allowed kinematic region is now:




2.) a.) $$R = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\sin\theta\\0\\\cos\theta\end{pmatrix} \qquad R\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\0\\-\sin\theta\end{pmatrix}$$

$$R\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$ this is the axis of the rotation

b.) $$\frac{d}{d\theta} R = \begin{pmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & 0 & 0 \\ -\cos\theta & 0 & -\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

so $R(\theta)$ satisfies

$$\frac{d}{d\theta} R(\theta) = -i \left[ \; J^2 \; \right] R(\theta)$$

where $$J^2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

the solution of this equation with initial condition $R(\theta=0)=1$

is $$R(\theta) = \exp\left[ -i J^2 \theta \right]$$

c.) In the spin-½ representation:

$$e^{-iJ^2\theta} = e^{-i \frac{\sigma^2}{2} \theta}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$(\sigma^2)^2 = 1$$

$$= \cos\frac{\theta}{2} - i\sigma^2 \sin\frac{\theta}{2} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

d) In the spin-1 representation:

$$\text{rotation by } \phi \text{ about } \hat{3} = R_1 = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{rotation by } \theta \text{ about } \hat{2} = R_2 = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\text{rotation by } -\phi \text{ about } \hat{3} : R_3 = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

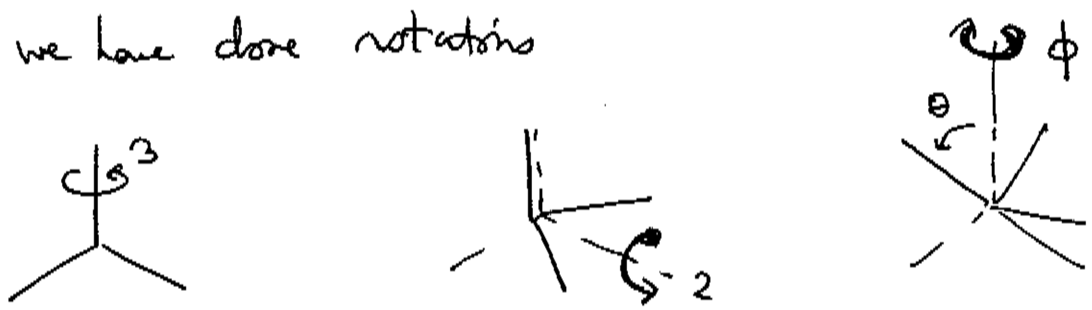
$$R_3 R_2 R_1 = \begin{pmatrix} \cos\phi & \sin\phi & \\ -\sin\phi & \cos\phi & \\ & & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & & \sin\theta \\ & 1 & \\ -\sin\theta & & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & \\ \sin\phi & \cos\phi & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\theta & \sin\phi & \cos\phi\sin\theta \\ -\sin\phi\cos\theta & \cos\phi & -\sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & \\ \sin\phi & \cos\phi & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\phi\cos\theta + \sin^2\phi & -\sin\phi\cos\phi\cos\theta + \sin\phi\cos\phi & \cos\phi\sin\theta \\ -\sin\phi\cos\phi\cos\theta + \sin\phi\cos\phi & \sin^2\phi\cos\theta + \cos^2\phi & -\sin\phi\sin\theta \\ -\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

what is this?

we have done rotations




It is obvious that the $\hat{3}$ axis should go into

rotat of $(\sin\theta, 0, \cos\theta)$

$$= (\cos\phi\sin\theta, -\sin\phi\sin\theta, \cos\theta)$$

but the rest is less obvious.

Try the spin-½ case:

$$R_1 = e^{-i\frac{\sigma^3}{2}\phi} = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix}$$

$$R_2 = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & +\cos\theta/2 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} e^{+i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$$

so

$$R_3R_2R_1 = \begin{pmatrix} \cos\theta/2\, e^{+i\phi/2} & -\sin\theta/2\, e^{+i\phi/2} \\ \sin\theta/2\, e^{-i\phi/2} & \cos\theta/2\, e^{-i\phi/2} \end{pmatrix}\begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \, e^{+i\phi} \\ \sin\theta/2 \, e^{-i\phi} & \cos\theta/2 \end{pmatrix}$$

now

$$e^{-i\vec{\theta}\cdot\vec{\sigma}/2} = \cos\frac{\theta}{2} - i\,\hat{\theta}\cdot\vec{\sigma}\,\sin\theta/2 \qquad \theta = |\vec{\theta}|$$

the above is :

$$R_3R_2R_1 = \cos\theta/2\cdot 1 - \cos\phi\,\sin\theta/2\begin{pmatrix}0&1\\-1&0\end{pmatrix} - i\sin\phi\,\sin\theta/2\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$= \cos\theta/2 - i\left[\sin\phi\,\sigma^1 + \cos\phi\,\sigma^2\right]\sin\theta/2$$

so this is a rotation about the axis

$$\hat{n} = \begin{pmatrix}\sin\phi\\ \cos\phi\\ 0\end{pmatrix}$$

by an angle $\theta$.

There are some relatively easy ways to check that p. 8 is also a rotation of $\theta$ about the axis $n$

For any rotation by $\theta$: $\operatorname{tr} R = \cos\theta + \cos\theta + 1 = 1 + 2\cos\theta$

For any rotation about $\hat{n}$: $R\hat{n} = \hat{n}$

Let's check

$$\mathrm{tr}(R_3R_2R_1) = \cos^2\phi\cos\Theta + \sin^2\phi + \sin^2\phi\cos\Theta + \cos^2\phi + \cos\Theta$$

$$= 2\cos\Theta + 1 \quad \checkmark$$

$$(R_3R_2R_1)\begin{pmatrix}\sin\phi\\ \cos\phi\\ 0\end{pmatrix}$$

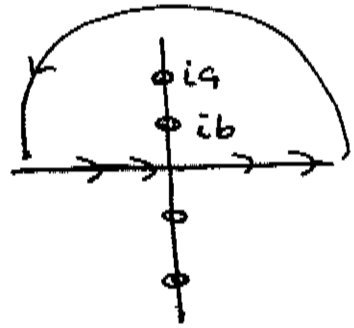
$$= \begin{pmatrix} (\cos^2\phi\cos\Theta + \sin^2\phi)\sin\phi + (-\sin\phi\cos\phi\cos\Theta + \sin\phi\cos\phi)\cos\phi \\ (-\sin\phi\cos\phi\cos\Theta + \sin\phi\cos\phi)\sin\phi + (\sin^2\phi\cos\Theta + \cos^2\phi)\cos\phi \\ (-\sin\Theta\cos\phi)\sin\phi + (\sin\Theta\sin\phi)\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \sin^3\phi + \sin\phi\cos^2\phi \\ \sin^2\phi\cos\phi + \cos^3\phi \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix}\sin\phi\\ \cos\phi\\ 0\end{pmatrix} = \hat{n} \quad \checkmark$$

so $R_3R_2R_1$ in the spin ½ and spin 1 cases are rotations about the same axis by the same angle.

3) a) $$I_1 = \int_0^\infty dx \, \frac{x^2}{(a^2+x^2)(b^2+x^2)}$$




$$= \frac{1}{2}\int_{-\infty}^{\infty} dx \, \frac{x^2}{(a^2+x^2)(b^2+x^2)}$$

$$= 2\pi i \cdot \frac{1}{2} \left\{ (\text{Residue at } x=ia) + (\text{Residue at } x=ib) \right\}$$

$$= \pi i \left\{ \frac{(ia)^2}{2ia\,(b^2-a^2)} + \frac{(ib)^2}{(a^2-b^2)\,(2ib)} \right\}$$

$$= \frac{\pi}{2(a^2-b^2)} \left\{ a-b \right\}$$

so

$$I_1 = \frac{\pi}{2} \frac{1}{(a+b)}$$

a more direct method: $$I_1 = \frac{1}{2}\int_{-\infty}^{\infty} dx \left( \frac{a^2}{a^2+x^2} - \frac{b^2}{b^2+x^2} \right) \frac{1}{(a^2-b^2)}$$

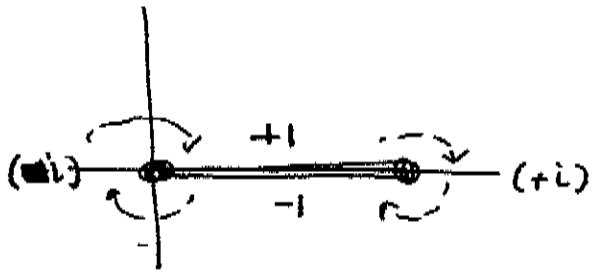
now $$\int_{-\infty}^{\infty} dx \, \frac{1}{a^2+x^2} = \frac{\pi}{a}$$

so $$I_1 = \frac{\pi}{2}(a-b)\frac{1}{a^2-b^2} = \frac{\pi}{2}\frac{1}{(a+b)} \checkmark$$

b.) To do this exercise, first look at the analytic structure of $\frac{1}{\sqrt{x(1-x)}}$. Define this so that

$$\frac{1}{\sqrt{x(1-x)}} = + \frac{1}{\sqrt{|x(1-x)|}} \quad \text{(real + positive)}$$

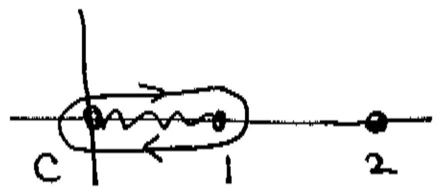
just above the real axis in the x plane. Then the phase of $\frac{1}{\sqrt{x(1-x)}}$ at other points is:



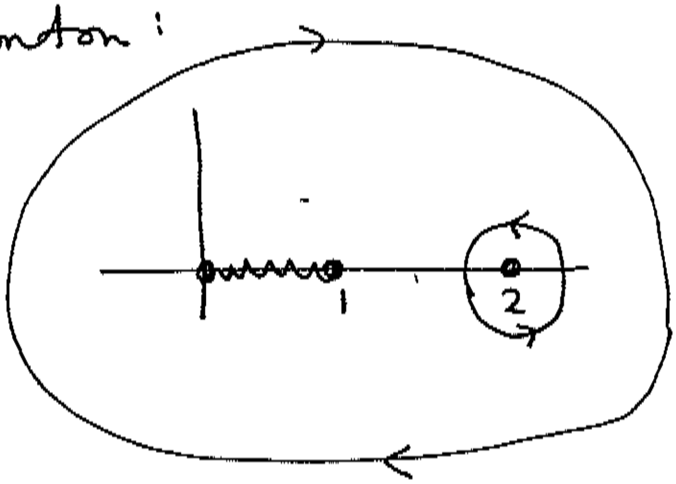

Notice that

$$I_2 = \int_0^1 dx \, \frac{1}{\sqrt{x(1-x)}} \, \frac{1}{x-2}$$

$$= \frac{1}{2} \int_C dx \, \frac{1}{\sqrt{x(1-x)}} \, \frac{1}{x-2}$$




Now deform the contour:



The big contour at $\infty$ gives $\int dx \, \frac{1}{x^2} \to 0$

The pole at $x=2$ gives

$$I_2 = \frac{1}{2} \, 2\pi i \left[ \frac{1}{\sqrt{x(1-x)}} \right]_{x=2}$$

$$= \frac{1}{2} \, 2\pi i \, (+i) \frac{1}{\sqrt{|2 \cdot 1|}}$$

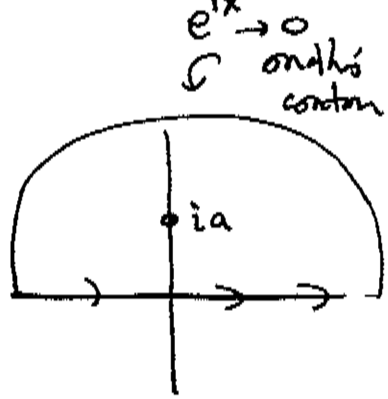
[ phase from p.13 ]

$$I_2 = -\frac{\pi}{\sqrt{2}}$$

c.) $$I_3 = \int_0^\infty dx \, \frac{x \sin x}{a^2 + x^2}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dx \, \frac{x}{a^2+x^2} \left( \frac{e^{ix} - e^{-ix}}{2i} \right)$$

$$= \int_{-\infty}^{\infty} dx \, \frac{x}{a^2+x^2} \, \frac{e^{ix}}{2i}$$




$$= \frac{2\pi i}{2i} \, \frac{ia}{2ia} \, e^{i(ia)}$$

$$I_3 = \frac{\pi}{2} e^{-a}$$

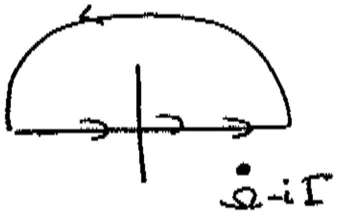
d.) The important point for this integral is that

$$e^{-i\omega t} \to 0 \quad \text{for} \quad \operatorname{Im}\omega \to \infty \quad \text{when} \quad t<0$$

$$\text{for} \quad \operatorname{Im}\omega \to -\infty \quad \text{when} \quad t>0$$

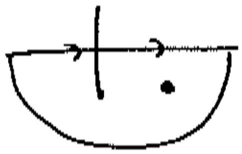
so

$t<0$



$$I_4 = 0$$

$t>0$



$$I_4 = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega - (\Omega - i\Gamma)}$$

$$= \frac{-2\pi i}{2\pi} e^{-it(\Omega - i\Gamma)}$$

so

$$I_4 = \begin{cases} 0 & t<0 \\ -i\, e^{-i\Omega t} e^{-\Gamma t} & t>0 \end{cases}$$