Physics 330 – Problem Set # 8

(due Wednesday, November 30)

1. Simplify the expression for 3-body phase space

$$\int d\Pi_3 = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3} (2\pi)^4 \delta^{(4)}(P - p_1 - p_2 - p_3) \tag{1}$$

in the CM frame: $P^\mu = (P, \vec{0})$. Do not assume any special values of the masses of the final particles; just take these to be m_1, m_2, m_3.

(a) Let $s = P^2$. Let

$$x_1 = \frac{2E_1}{\sqrt{s}} \qquad x_2 = \frac{2E_2}{\sqrt{s}} \qquad x_3 = \frac{2E_3}{\sqrt{s}} \tag{2}$$

Show that $x_1 + x_2 + x_3 = 2$. We can take any two of these variables as independent.

(b) Show that the invariants $m_{ij}^2 = (p_i + p_j)^2$ are uniquely determined as functions of the x_i: For example,

$$m_{12}^2 = s(1 - x_3) + m_3^2 \tag{3}$$

(c) Show that, in the CM system, $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$ lie in a plane (the 'event plane'). Show that the cosine of the angle between any two particles in this plane is uniquely determined as a function of the x_i. The final state then has 5 degrees of freedom– two of the x_i and three Euler angles for the orientation of the event plane. Since $9 - 4 = 5$, this number is right.

(d) Integrate over the orientation of the event plane and derive an expression for 3-body phase space as an integral over x_1 and x_2. You should find

$$\int d\Pi_3 = \frac{s}{128\pi^3} \int dx_1 dx_2 \tag{4}$$

The region of integration is somewhat subtle to determine. You worked out the cases of 3 massless and 1 massive and 2 massless particles in Problem Set 1.

(e) Show that the result (4) implies the statement, 'Three body phase space is flat as a function of m_{12}^2 and m_{13}^2.'

This result suggests a way to study reactions with 3-body final states, for example $B^+ \to K^+ K^- \pi^+$: Perform a scatter plot of the data in the plane of $m^2(K^+K^-)$ *vs.* $m^2(\pi^+ K^-)$. This is called the 'Dalitz plot'. An accumulation of points in this plot on a line parallel to one of the axes indicates a 2-particle resonance in the final state.

2. As a simple model of the strong interactions, assume that quarks interact through exchange of a massless vector boson G with the coupling

$$\Delta\mathcal{L} = g\overline{q}\gamma^\mu q G_\mu \tag{5}$$

This is called the 'Abelian vector gluon' theory. Compute the cross section of gluon emission in e^+e^- annihilation,

$$\frac{d\sigma}{dx_1 dx_2}(e^+e^- \to q\overline{q}G) \tag{6}$$

where x_1, x_2 are the variables (6) for the quark and antiquark respectively. Assume that the quark has electric charge Q_f and 3 colors. Define $\alpha_g = g^2/4\pi$. Ignore the mass of the electron and the mass of the quark. G couples only to quarks and antiquarks, not to electrons.

(a) Draw the Feynman diagrams for $e^+e^- \to q\overline{q}G$ and evaluate the scattering amplitudes. Square and find the polarization sum and average as a product of two traces.

(b) Now we have to integrate over angles. A sneaky but very convenient way to do this is to hold the event plane fixed and average over the orientation of the initial state. Let the electron and positron momenta be p^μ, $\overline{p}^\mu$, respectively. By writing out the components of the tensor and averaging over angles (or by any other method you can think of), show that

$$\langle p^\mu \overline{p}^\nu + p^\nu \overline{p}^\mu - g^{\mu\nu} p \cdot \overline{p} \rangle = -\frac{1}{3}\left(g^{\mu\nu}P^2 - P^\mu P^\nu\right) \tag{7}$$

where $P = p + \overline{p}$.

(c) Carry out the remaining trace computation. Current conservation might be useful to simplify the computation. You might recognize the trace as (a crossing of) one that you encountered already in the last problem of the previous problem set. Derive the result

$$\frac{d\sigma}{dx_1 dx_2}(e^+e^- \to q\overline{q}G) = \frac{4\pi\alpha^2}{3s} \cdot 3Q_f^2 \cdot \frac{\alpha_g}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \tag{8}$$

This same result is correct (to leading order) in the real theory of the strong interactions, QCD, with the replacement

$$\alpha_g \to \frac{4}{3}\alpha_s \tag{9}$$