Physics 330 – Problem Set # 4

(due Wednesday, October 26)

1. Using the transformation properties of $\psi(x)$ and $\overline{\psi}(x)$ under P, T, and C given in eqs. (3.126), (3.128), (3.139), (3.140), (3.145), (3.146) of Peskin and Schroeder, work out explicitly the transformations of the 16 Dirac fermion bilinears under C, P, and T. In this way, verify all of the entries in the table on p. 71 of Peskin and Schroeder.

2. (a) Show that the state

$$|\Phi(K)\rangle = \sqrt{2E_K} \int \frac{d^3k}{(2\pi)^3} \tilde{\phi}(\vec{k}) a^{\dagger s}_{K/2+k} b^{\dagger t}_{K/2-k} |0\rangle \tag{1}$$

is a correctly normalized bound state of a fermion with $S^3 = s$ and an antifermion with $S^3 = t$, assuming that

$$\int \frac{d^3k}{(2\pi)^3} \left|\tilde{\phi}(\vec{k})\right|^2 = 1 \tag{2}$$

and that $E_K^2 = M^2 + |\vec{K}|^2$, where M is the mass of the state. How is $\tilde{\phi}(\vec{k})$ related to the Schrödinger wavefunction $\phi(\vec{x})$ of the bound state? For the rest of this problem, assume that $\tilde{\phi}$ is S-wave: $\tilde{\phi}(\vec{k}) = \tilde{\phi}(|\vec{k}|)$.

(b) Let s and t both be spin up ($S^3 = +\frac{1}{2}$). For a bound state at rest, $\vec{K} = 0$, compute

$$\langle 0| \overline{\psi}\gamma^\mu\psi(0) |\Phi(K)\rangle \quad \text{and} \quad \langle 0| \overline{\psi}\gamma^\mu\gamma^5\psi(0) |\Phi(K)\rangle \ , \tag{3}$$

where $\psi(x)$ is the Dirac quantum field. It suffices to work in the leading approximation for non-relativistic fermions:

$$u^s(p) \approx \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \qquad v^s(p) \approx \sqrt{m} \begin{pmatrix} \xi^{-s} \\ -\xi^{-s} \end{pmatrix} \ . \tag{4}$$

Which matrix elements are nonzero? Why?

(c) Let s and t be combined into a spin-0 state. For a bound state at rest, $\vec{K} = 0$, compute

$$\langle 0| \overline{\psi}\psi(0) |\Phi(K)\rangle \quad \text{and} \quad \langle 0| \overline{\psi}\gamma^5\psi(0) |\Phi(K)\rangle \tag{5}$$

again using the leading nonrelativistic approximation. Which matrix elements are nonzero? Why?

3. The action for the electromagnetic field coupled to a c-number current $j^\mu(x)$ is written in relativistic notation as

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu \right) \ , \tag{6}$$

where $A^\mu = (\varphi, \vec{A})$ is the vector potential and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7}$$

is the field strength.

(a) Work out the relativistic form of the variational equation of motion for A_μ.

(b) Define $E^i = F^{i0}$, $\epsilon^{ijk} B^k = -F^{ij}$. Show that the equations for E and B are the familiar Maxwell equations (in 'rationalized Heaviside units' with $\epsilon_0 = \mu_0 = c = 1$).

(c) Compute the canonical momentum conjugate to A^μ

$$\frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} \tag{8}$$

Show that A^0 has no conjugate momentum. This is related to the fact that the corresponding equation of motion

$$\vec{\nabla} \cdot \vec{E} = j^0 \tag{9}$$

does not contain a time derivative. Coordinates without conjugate momenta occur in equations with a gauge invariance, that is, with the freedom to redefine variables by an arbitrary function of t or x^μ in a way that does not change the physics. For electrodynamics, show that the change of variables

$$A'^\mu = A^\mu - \partial^\mu \alpha(x) \tag{10}$$

does not affect Maxwell's equations.

(d) The Hamiltonian formalism for a system with gauge invariance typically requires specifying a gauge. For electrodynamics, a convenient choice is Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$. Then also $(d/dt)(\vec{\nabla} \cdot \vec{A}) = 0$. Show that this allows one to solve for A^0 in terms of the source current j^0.

(e) With A^0 and $\vec{\nabla} \cdot \vec{A}$ eliminated, this system now has only two degrees of freedom per point, the two components of A^i that satisfy $\vec{\nabla} \cdot \vec{A} = 0$. At this point, it is more convenient to go to Fourier space and constrain $\vec{A}(\vec{k})$ by

$$\vec{k} \cdot \vec{A}(\vec{k}) = 0 \tag{11}$$

The conjugate momentum should obey a similar constraint. An appropriate form for the canonical commutation relation is then

$$[A^i(\vec{k}), \Pi^j(-\vec{k}')] = i\left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2}\right)(2\pi)^3\delta(\vec{k} - \vec{k}') \tag{12}$$

Find a representation of this commutation relation in which $\vec{A}$ has the form

$$\vec{A}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2p}}e^{i\vec{p}\cdot\vec{x}}\sum_a\left(\vec{\epsilon}_a(\vec{p})a_{\vec{p}a} + \vec{\epsilon}_a^{\,*}(-\vec{p})a^\dagger_{-\vec{p}a}\right) \tag{13}$$

where $\vec{\epsilon}_a(\vec{p})$ is a transverse unit vector, $\vec{p}\cdot\vec{\epsilon}_a(\vec{p}) = 0$, and $a = 1, 2$.

(f) What is the relation between $\vec{\Pi}$ and $\vec{E}$?

(g) Show that the representation (12) diagonalizes

$$H = \int d^3x\left[\frac{1}{2}E^2 + \frac{1}{2}B^2\right] \tag{14}$$

The result of these manipulations is a theory of transverse quantum photons and c-number Coulomb potentials. If j^0 is replaced by a quantum operator, we would obtain a fully quantum theory of electromagnetism. The theory is not manifestly relativistically invariant, and it is not manifestly local (since, for example, (11) is nonlocal when transformed back to real space). In class, I will give another description of electrodynamics which is manifestly relativistically invariant and local but which can contain negative probabilities. It is true but nontrivial to show that these descriptions are equivalent in a subspace of their Hilbert spaces; in this subspace, all bad features disappear.