Physics 330 – Problem Set # 3

(due Wednesday, October 19)

1. Peskin and Schroeder, problem 3.2. Use the fact that the $u(p)$ satisfy $(\gamma^\mu p_\mu)u(p) = mu(p)$.

2. Peskin and Schroeder, problem 3.4.

3. Peskin and Schroeder, problem 3.5.

Problems 2 and 3 illustrate that it is advantageous to consider even the *classical* spin-$\frac{1}{2}$ field as an anticommuting object. Classical anticommuting numbers ('Grassmann numbers') a, b satisfy

$$ab = -ba \ , \qquad a^2 = b^2 = 0 \ , \qquad (ab)^\dagger = b^\dagger a^\dagger = -a^\dagger b^\dagger \ . \tag{1}$$

The last part of Problem 2, concerning the quantization of the Majorana fermion theory, is tricky. Here are some hints:

(1) The canonical anticommutation relations should be

$$\{\chi_a(\vec{x}), \chi_b^\dagger(\vec{y})\} = \delta_{ab}\delta(\vec{x} - \vec{y}) \tag{2}$$

The Hamiltonian is

$$H = \int d^3x \Big\{ \chi^\dagger i\vec{\sigma}\cdot\vec{\nabla}\chi - \frac{i}{2}m(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*) \Big\} \tag{3}$$

Show that this Hamiltonian leads to the correct Heisenberg equation of motion for $\chi(x)$.

(2) Since fields will be expanded in terms of the solutions of the Majorana equation, the first thing to do is to find a representation of these solutions. Show that expressions that look like the upper two components of the usual u and v solutions of the Dirac equation:

$$\chi^{(+)} = \sqrt{p\cdot\sigma}\xi e^{-ip\cdot x} \qquad \chi^{(-)} = \sqrt{p\cdot\sigma}(-i\sigma^2)\xi^* e^{ip\cdot x} \ , \tag{4}$$

where ξ is a 2-component spinor, solve

$$i\overline{\sigma}\cdot\partial\chi^{(+)} = im\sigma^2\chi^{(-)*} \tag{5}$$

and the same equation with $(+)$ and $(-)$ interchanged. This motivates the representation:

$$\chi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \Big\{ \sqrt{p\cdot\sigma}\xi^s a^s_{\vec{p}} + \sqrt{\hat{p}\cdot\sigma}(-i\sigma^2)\xi^{s*} a^{s\dagger}_{-\vec{p}} \Big\} e^{i\vec{p}\cdot\vec{x}} \ , \tag{6}$$

where $\hat{p} = (E_p, -\vec{p})$. Show that this representation satisfies the canonical anticommutation relations and diagonalizes H.