Physics 330 – Problem Set # 2

(due Wednesday, October 12)

The first two problems are close to Peskin and Schroeder, problem 2.2. The changes in notation are intentional; hopefully, they will make the exercise more clear.

1. Consider the field theory of a free complex scalar field. The action of the theory is

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right) \tag{1}$$

where $\phi(x)$ is a complex-valued field and m is real-valued. It is easiest to deal with this field by treating $\phi(x)$ and $\phi^*(x)$ as the independent variables rather than by using the real and imaginary parts of $\phi(x)$.

(a) By varying S, show that $\phi(x)$ and $\phi^*(x)$ satisfy the Klein-Gordon equation.

(b) Show that the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ are

$$\pi^*(x) = \dot{\phi}^*(x) \qquad \pi(x) = \dot{\phi}(x) \ , \tag{2}$$

respectively. Show that the Hamiltonian of the theory is

$$H = \int d^3x \left(\pi^* \pi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + m^2 \phi^* \phi \right) \tag{3}$$

(c) Using the canonical commutation relations

$$[\phi(\vec{x}), \pi^*(\vec{y})] = i\delta(\vec{x} - \vec{y}) \qquad [\phi^*(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x} - \vec{y}) \tag{4}$$

(with all other pairs commuting), show that the Heisenberg equation of motion for $\phi(x)$ is the Klein-Gordon equation.

(d) Introducing appropriate representations for the operators $\phi(x)$, $\phi^*(x)$, $\pi(x)$, $\pi^*(x)$ in terms of creation and annihilation operators, diagonalize H. Show that two sets of creation and annihilation operators are needed to represent the canonical commutation relations. Show that these create particles of equal mass. Argue that the corresponding excitations can be considered particle and antiparticle.

(e) Consider the quantity

$$Q = -i \int d^3x \left(\pi^* \phi - \phi^* \pi \right) \tag{5}$$

Express Q in terms of creation and annihilation operators. Show that Q commutes with H. Show explicitly that single-particle eigenstates of H are eigenstates of Q. How do you interpret the eigenvalues?

2. Generalize the preceding problem to the case of two complex Klein-Gordon fields ϕ_a, $a = 1, 2$ with equal mass m.

   (a) Write down an appropriate Hamiltonian and canonical commutation relations, and show that $\phi_a(x)$ obeys the Klein-Gordon equation as its equation of motion.

   (b) Show that the charge
   $$Q = -i \int d^3x \, (\pi_a^* \phi_a - \phi_a^* \pi_a) \tag{6}$$
   commutes with H. (Sums over repeated indices should be understood.)

   (c) Show that, if σ_{ab}^j are the Pauli sigma matrices, the charges
   $$Q^j = -\frac{i}{2} \int d^3x \left(\pi_a^* \sigma_{ab}^j \phi_a - \phi_a^* \sigma_{ab}^j \pi_b \right) \tag{7}$$
   commute with H. Compute
   $$[Q^j, Q^k] \tag{8}$$

3. Peskin and Schroeder, problem 3.1.