Physics 330 – Problem Set # 1

(due Wednesday, October 5)

The purpose of this problem set is to help you review some items of mathematics and physics that will be used in the course. If it turns out that this is not review, and that in fact you have never done these sorts of calculations before, please take special care that you are familiar with these methods.

1. Relativistic kinematics:

   (a) A particle of mass M decays to a particle of mass m and a massless particle. Find the energies and momenta of the two final particles.

   (b) A particle of mass M decays to two particles, one with mass m_1 and the other with mass m_2. Find the energies and momenta of the two final particles.

   (c) A particle of mass M decays to three massless particles. Let variables $x_i = 2E_i/M$ parametrize the energies of the final particles. Note that $\sum_i x_i = 2$. Find the kinematically allowed region for (x_1, x_2).

   (d) A particle of mass M decays to two massless particles (particles 1 and 2) and one particle of mass m (particle 3). Let variables $x_i = 2E_i/M$ parametrize the energies of the final particles. Again, $\sum_i x_i = 2$. Find the kinematically allowed region for (x_1, x_2). Show that the boundary in this case is formed from a straight line and a hyperbola.

2. Rotations and spin:

   (a) Write the 3×3 rotation matrix for a rotation by θ about the $\hat{2}$ (y) axis that carries the vector $(0,0,1)$ into $(\sin\theta, 0, \cos\theta)$.

   (b) Show that this matrix is represented by $R = \exp[-i\theta J^2]$, where $J^2_{13} = -J^2_{31} = i$ and all other elements are 0.

   (c) The corresponding rotation matrix in the spin-$\frac{1}{2}$ representation is generated by $J^2 = \sigma^2/2$. Write this matrix explicitly as a 2×2 matrix.

   (d) Compute the rotation matrix for the following sequence of operations: (1) rotate about $\hat{3}$ by ϕ; (2) rotate about $\hat{2}$ by θ; (3) rotate about $\hat{3}$ by $-\phi$. Perform this computation both in the vector (spin-1) and the spin-$\frac{1}{2}$ representation. Show that the same rotation is produced in the two cases, that is, that the two matrices represent a rotation about the same axis by the same angle.

3. Complex variables and contour integration:

(a) Evaluate

$$I_1 = \int_0^\infty dx \, \frac{x^2}{(a^2 + x^2)(b^2 + x^2)} \tag{1}$$

(b) Evaluate

$$I_2 = \int_0^1 dx \, \frac{1}{\sqrt{x(1-x)}\,(x-2)} \tag{2}$$

(c) Evaluate

$$I_3 = \int_0^\infty dx \, \frac{x \sin x}{(a^2 + x^2)} \tag{3}$$

(d) Evaluate

$$I_4 = \int_{-\infty}^\infty \frac{d\omega}{2\pi} \, \frac{e^{-i\omega t}}{\omega - (\Omega - i\Gamma)} \ , \tag{4}$$

where Ω and Γ are real and positive. Consider the cases $t > 0$ and $t < 0$ separately.