Physics 330 – Final Exam

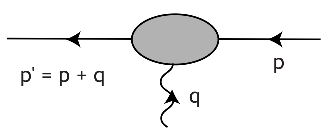
This exam is due on Wednesday, December 10. Please return the exam to Mariel Haag in Varian 336 by 3:00 pm sharp. If you have any questions about the exam, please contact me at mpeskin@slac.stanford.edu or 926-3250. If errata are reported, I will announce them on the course Web page.

Please do not collaborate on this exam. Please return the exam in a blue book (or multiple blue books) with the honor code acknowledgement signed.

The exam is worth a total of 100 points (plus a possible 5 points extra credit). The distribution of points is indicated below.

The problem deals with the lepton number violating processes $\mu \to e\gamma$, $\tau \to \mu\gamma$, $\tau \to e\gamma$. These decay processes have never been observed. In the 'Standard Model' of particle physics, each species of lepton carries a conserved quantum number and cannot convert to another species. However, most models of physics beyond the Standard Model allow some mechanism for these flavor-changing lepton decays, and so it is interesting to look for these decays experimentally.

a. (20 points) We will see below that a flavor-changing lepton decay such as $\mu \to e\gamma$ is mediated by the 'transition dipole matrix element':



$$= -ie\overline{u}(p')\left[i\sigma^{\mu\nu}q_\nu \overline{F}_2(q^2)\right]u(p) \tag{1}$$

Compute the decay rate for $\mu \to e\gamma$ in terms of the coefficient $\overline{F}_2(0)$. You may set $m_e = 0$ throughout this problem set, except in part (j).

b. (5 points) The typical decay of the muon is by the weak-interaction process

$$\mu^- \to \nu_\mu e^- \overline{\nu}_e \tag{2}$$

The rate of this decay is:

$$\Gamma(\mu) = \frac{\alpha_w^2 m_\mu^5}{384\pi m_W^4} = (2.2 \times 10^{-6}\ \text{sec})^{-1} \tag{3}$$

where $\alpha_w = 1/29.6$ is the coupling strength of the weak interaction, $m_\mu = 0.1057$ GeV is the mass of the muon, and $m_W = 80.4$ GeV is the mass of the W boson. Similarly, the rate of weak-interaction decay of the τ is given by

$$\Gamma(\tau) = 5.3\frac{\alpha_w^2 m_\tau^5}{384\pi m_W^4} = (2.9 \times 10^{-13}\ \text{sec})^{-1} \tag{4}$$

where $m_\tau = 1.777$ GeV and the coefficient in front equals $(1+1+3.3)$ to account fir the decay of the τ^- to e^-, to μ^-, and to quarks with 3 colors and a 10% enhancement from QCD corrections. (These formulae are not hard to derive using methods from Physics 330, but you may just take them as given.) To compare to these formulae, parametrize the form factor $\overline{F}_2$ in the transition dipole moment as

$$\overline{F}_2(0) = G\frac{\alpha}{4\pi}\frac{m_\ell}{M^2} \tag{5}$$

where m_ℓ is the mass of the decaying lepton, M is the mass of some new particle associated with the flavor violation, and G is a dimensionless constant. The current limits on the branching ratios for flavor changing decays are (at the 90% CL):

$$\begin{aligned} BR(\mu \to e\gamma) &< 1.2 \times 10^{-11} \\ BR(\tau \to e\gamma) &< 2.7 \times 10^{-6} \\ BR(\tau \to \mu\gamma) &< 3.2 \times 10^{-7} \end{aligned} \tag{6}$$

What are the corresponding limits on the values of G, assuming that $M = 200$ GeV (beyond the reach of current experiments)? Which is the strongest constraint if the predictions for G are the same for the three processes? What if the predictions for G depend on the lepton mass and are proportional to m_ℓ^2?

c. (10 points) The last result in (6) is a new result from the BELLE B-factory experiment. The analysis made use of the reaction $e^+e^- \to \tau^+\tau^-$ using 86.3 fb^{-1} of data at $E_{cm} =$ 10.58 GeV. The definition of 1 fb^{-1} is that it is the number of collisions needed to produce 1 event for a process whose cross section is 1 fb $= 1 \times 10^{-15}$ barn. How many $\tau^+\tau^-$ pairs were produced in this experiment? (The BaBar and BELLE experiments should eventually each record more than 1000 fb^{-1} of data, leading to very strong constraints on the radiative τ decays.)

d. (5 points) With this introduction, let's compute the transition dipole moment in a simple model. Write down the Lagrangian for the quantum electrodynamics of electrons and muons (with fields e, μ, A^μ). Add to this:

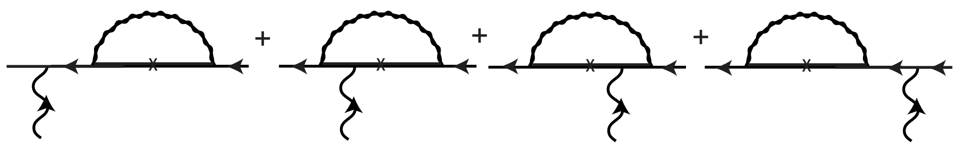
$$\begin{aligned} \mathcal{L} &= (D_\nu\phi_e)^* D^\nu\phi_e - m^2\phi_e^*\phi_e + (D_\nu\phi_\mu)^* D^\nu\phi_\mu - m^2\phi_\mu^*\phi_\mu - \delta m^2(\phi_e^*\phi_\mu + \phi_\mu^*\phi_e) \\ &\quad + \overline{b}(i\not{\partial} - m_b)b + g\left(\overline{e}b\phi_e + \overline{\mu}b\phi_\mu + \phi_e^*\overline{b}e + \phi_\mu^*\overline{b}\mu\right) \end{aligned} \tag{7}$$

In this Lagrangian, ϕ_e and ϕ_μ are complex scalar fields carrying electron and muon number, respectively, and b is an electrically neutral Dirac fermion. The coupling constant of the new fields is taken to be g, which you should consider to be of the order of e. The two scalar fields are given the same mass for simplicity. The covariant derivative is $D_\nu = \partial_\nu + ieA_\nu$. In thinking about this Lagrangian, imagine that m and m_b are both much larger than 100 GeV, while $m_\mu = 0.106$ GeV.

This Lagrangian is a simplified version of a supersymmetric theory of new particle physics interactions. The scalar fields ϕ_e and ϕ_μ represent the supersymmetric partners of the electron and muon. The fermion b is the supersymmetric partner of the photon. In a supersymmetric theory, $g = \sqrt{2}e$.

Write the Feynman rules for this theory, treating the δm^2 term as a perturbation that gives rise to a 2-point vertex.

e. (10 points) In the theory (7), the matrix element for the process $\mu \to e\gamma$ is given by the following set of Feynman diagrams:



(8)

with μ on the right, e on the left, and b and $\phi_\mu \to \phi_e$ in the loop in each diagram. Write the values of these diagrams, without doing any of the integrals or Dirac algebra.

Notice that, since $m_\mu \neq m_e$, the diagrams with self-energy-type integrals are not infinite and should be treated as ordinary contributions to the S-matrix element. On the other hand, there are no contributions to $(Z_2 - 1)$ proportional to δm^2, so the sum of diagrams shown in (8) is the whole calculation.

f. (10 points) Show that the expression you wrote in part (e) satisfies the Ward identity:

$$q_\mu (\text{sum of diagrams})^\mu = 0 \ . \tag{9}$$

Use the fact that the initial μ and the final e are on shell, so that $(\not{p} - m_\mu)u(p) = 0$, $(\not{p}' - m_e)u(p') = 0$.

g. (5 points) Show that the Gordon identity for this kinematic situation is

$$(m_e + m_\mu)\overline{u}(p')\gamma^\nu u(p) = \overline{u}(p') \left[(p' + p)^\nu + i\sigma^{\nu\sigma} q_\sigma\right] u(p) \tag{10}$$

Using this identity, the sum of diagrams in (e) can be gathered into the form

$$-ie\overline{u}(p') \left[\gamma^\mu F_1(q^2) + q^\mu F_3(q^2) + i\sigma^{\mu\nu} q_\nu \overline{F}_2(q^2)\right] u(p) \ . \tag{11}$$

Using the Ward identity, find a relation between the form factors F_1 and F_3. In particular, show that $F_1(q^2)$ vanishes at $q^2 = 0$ if $F_3(q^2)$ has a smooth and nonsigular limit as $q^2 \to 0$.

h. (15 points) Compute the Feynman diagrams shown in (8). Notice that all integrals are UV-finite. Write expressions for $F_1(q^2)$, $F_3(q^2)$, and $\overline{F}_2(q^2)$ as integrals over Feynman parameters. Show that all three form factors have nonsingular limits as $q^2 \to 0$ and as $m_\mu \to 0$. (Remember that $m_\mu \ll m, m_b$.) You may ignore m_μ in the rest of the calculation (except in part (j)).

i. (5 points) Show explicitly that $F_1(q^2) = 0$ at $q^2 = 0$.

j. (5 points extra credit) Show the relation between F_1 and F_3 found in (g) holds explicitly without first taking the limits $m_\mu \to 0$, $m_e \to 0$, $q^2 \to 0$.

k. (5 points) Show that F_1 and F_3 do not contribute to the amplitude for $\mu \to e\gamma$.

l. (10 points) Compute $\overline{F}_2(0)$ in the limit $m_\mu \to 0$. Notice that it is of the general form of (5), except that m_ℓ is replaced by m_b. In a realistic theory of supersymmetry, this defect would be remedied. In a typical model $\delta m^2/m^2$ of the order of $\alpha/4\pi$, but in some models this quantity is parametrically larger for decays of the τ.

References: This exam is open-book. If you make strong use of any reference other than the class textbook and notes, please give the reference. Be warned: There are many papers on $\mu \to e\gamma$, but none are simple to follow. It is probably easier just to do the problem from scratch.

Self-evaluation: To record a satisfactory performance on this exam, please complete at least through part (e). Prospective theorists should slog through to the end.