Physics 330 – Final Exam

Solutions

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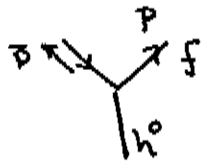
a.) 

$$= -i\,\frac{m_f}{v}$$

$$i\mathcal{M} = -i\,\frac{m_f}{v}\;\bar{u}(p)\,v(\bar{p})$$

$$\sum_{\text{spin}} |\mathcal{M}|^2 = \left(\frac{m_f}{v}\right)^2 \operatorname{tr}\,(\not{p}+m_f)(\not{\bar{p}}-m_f)$$

$$= 4\left(\frac{m_f}{v}\right)^2 \left(p\cdot\bar{p} - m_f^2\right)$$

now $(p+\bar{p})^2 = m_h^2 = 2m_f^2 + 2\,p\cdot\bar{p}$

$$= 2\left(\frac{m_f}{v}\right)^2 \left(m_h^2 - 4m_f^2\right)$$

$$\Gamma(h\to f\bar{f}) = \frac{1}{2m_h}\,\frac{(2p/m_h)}{8\pi}\cdot 2\left(\frac{m_f}{v}\right)^2 m_h^2\left(1-\frac{4m_f^2}{m_h^2}\right)$$

$$= \frac{m_h}{8\pi}\left(\frac{m_f}{v}\right)^2\left(1-\frac{4m_f^2}{m_h^2}\right)^{3/2}$$

for quarks there is an extra factor $\underline{3}$ for color.

note: 1. Because h^0 is a scalar, there is no spin-averaging. We just sum over spins in the final state.

2. Because h^0 has $P=+1$ but an $f\bar{f}$ state has $P = (-1)\cdot(-1)^L$, h^0 decays into a state with $L=1$. This explains the extra factor of $|\vec{p}|^2 = \left(\frac{m_h}{2}\right)^2\left(1-\frac{4m_f^2}{m_h^2}\right)$

b.) For $m_h = 120$ GeV $\quad \frac{m_h}{8\pi}\left(\frac{1\,\text{GeV}}{v}\right)^2 = 79$ keV

then

$h^0 \to c\bar{c} \quad \times (1.2)^2 \times \left(1-\frac{4m_c^2}{m_h^2}\right)^{3/2} \cdot 3 = 340$ keV

$h^0 \to t\bar{t} \quad$ forbidden by energy conservation

$h^0 \to b\bar{b} \quad \times (4.2)^2 \cdot \left(1-\frac{4m_b^2}{m_h^2}\right)^{3/2} \cdot 3 = 4150$ keV

$h^0 \to \mu^+\mu^- \quad \times (0.1)^2 \cdot \left(1-\frac{4m_\mu^2}{m_h^2}\right)^{3/2} = 0.87$ keV

$h^0 \to \tau^+\tau^- \quad \times (1.77)^2 \cdot \left(1-\frac{4m_\tau^2}{m_h^2}\right)^{3/2} = 247.$ keV

color factor for quarks only

the sum of these values is:

$$\Gamma(h^0) = 4.7 \text{ MeV}$$

decay rate $\quad (\tau)^{-1} = (\hbar/\Gamma)^{-1} = (1.4\times 10^{-22}\,\text{sec})^{-1} = 7.1\times 10^{+21}/\text{sec}$

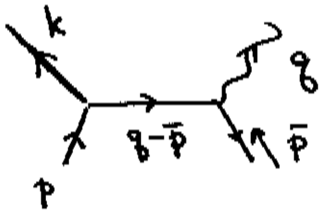
the branching fraction to $f\bar{f}$ is

$$BR(h \to f\bar{f}) = \frac{\Gamma(h \to f\bar{f})}{\Gamma(h)}$$

$c\bar{c}$	$t\bar{t}$	$b\bar{b}$	$\mu^+\mu^-$	$\tau^+\tau^-$
7%	0	88%	1.8×10^{-4}	5%

A more complete study of h° decays would include additional modes $h^\circ \to WW^*$, $h^\circ \to ZZ^*$, $h^\circ \to gg$ and a better accounting of QCD corrections. See The Higgs Hunter's Guide by Gunion, Haber, Kane, & Dawson.

c.)



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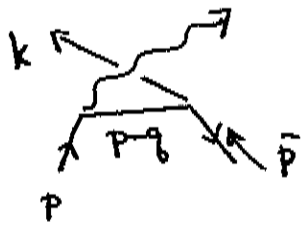


$$i\mathcal{M} = (+iQ_f e)\left(-i\frac{m_f}{v}\right) \varepsilon^{*\mu}(q)$$

$$\cdot\ \bar{u}(\bar{p}) \left[\gamma_\mu \frac{i(\not{q} - \not{\bar{p}})}{(q-\bar{p})^2} + \frac{i(\not{p} - \not{q})}{(p-q)^2} \gamma_\mu \right] u(p)$$

$q^2 = 0 \qquad k^2 = m_h^2 \qquad p^2 = \bar{p}^2 = 0$ since we ignore the quark mass

$$(q-\bar{p})^2 = -2q\cdot\bar{p} \qquad (q-p)^2 = -2q\cdot p$$

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 = \frac{1}{4}e^2 Q_f^2 \left(\frac{m_f}{v}\right)^2 (-g_{\mu\lambda})$$

$$\cdot \operatorname{tr}\left[\bar{\not p}\left(\frac{\gamma_\mu(\bar{\not p}-\not q)}{2\bar{p}q} - \frac{(\not p-\not q)\gamma_\mu}{2p\cdot q}\right)\not p\left(\frac{(\bar{\not p}-\not q)\gamma_\lambda}{2\bar{p}\cdot q} - \frac{\gamma_\lambda(\not p-\not q)}{2p\cdot q}\right)\right]$$

$$= -\frac{1}{4}e^2 Q_f^2\left(\frac{m_f}{v}\right)^2 \cdot \left\{\frac{\mathrm{I}}{(2\bar{p}\cdot q)^2} - \frac{\mathrm{II}}{2\bar{p}\cdot q\, 2p\cdot q} - \frac{\mathrm{III}}{2p\cdot q\, 2\bar{p}\cdot q} + \frac{\mathrm{IV}}{(2p\cdot q)^2}\right\}$$

$$\mathrm{I} = \operatorname{tr} \bar{\not p}\gamma_\mu(\bar{\not p}-\not q)\not p(\bar{\not p}-\not q)\gamma^\mu$$

$$= -2\operatorname{tr}\bar{\not p}(\bar{\not p}-\not q)\not p(\bar{\not p}-\not q)$$

$$= -2\cdot 4\,(\bar{p}\cdot(\bar{p}-q)\, p\cdot(\bar{p}-q)\cdot 2 - \bar{p}\cdot p\,(\bar{p}-q)^2)$$

$$= -2\cdot 4\cdot(\cancel{-2\bar{p}\cdot q\; p\cdot\bar{p}} + 2\bar{p}\cdot q\, p\cdot q - \cancel{\bar{p}\cdot p(-2\bar{p}\cdot q)})$$

$$= -4\cdot 2\bar{p}\cdot q\cdot 2p\cdot q$$

$$\mathrm{II} = \operatorname{tr}\bar{\not p}\gamma_\mu(\bar{\not p}-\not q)\not p\gamma^\mu(\not p-\not q)$$

$$= 4\operatorname{tr}\bar{\not p}(\bar{p}-q)\cdot p\,(\not p-\not q)$$

$$= 4\cdot 4\cdot \bar{p}\cdot(p-q)\; p\cdot(\bar{p}-q)$$

$$= 4\,(2p\cdot\bar{p} - 2\bar{p}\cdot q)(2p\cdot\bar{p} - 2p\cdot q)$$

$$\mathrm{III} = \mathrm{tr}\, \bar{\not{p}}\, (\not{p} - \not{q})\, \gamma^\mu\, \not{p}\, (\bar{\not{p}} - \not{q})\, \gamma_\mu$$

$$= 4\ \mathrm{tr}\ \bar{\not{p}} \cdot (\not{p} - \not{q}) \quad p \cdot (\bar{p} - q)$$

$$= 4\ (2\bar{p} \cdot p - 2\bar{p} \cdot q)\ (2\bar{p}p - 2p \cdot q)$$

$$\mathrm{IV} = \mathrm{tr}\left[\bar{\not{p}}\ (\not{p} - \not{q})\, \gamma^\mu\, \not{p}\, \gamma_\mu\, (\not{p} - \not{q})\right]$$

$$= (-2)\ \mathrm{tr}\ \bar{\not{p}}\, (\not{p} - \not{q})\, \not{p}\, (\not{p} - \not{q})$$

$$= -2 \cdot 4\left[\bar{p} \cdot (p - q)(-p \cdot q) \cdot 2 \ - \ \bar{p} \cdot p\, (p - q)^2\right]$$

$$= -2 \cdot 4\left[\cancel{\bar{p} \cdot p\, (-2p \cdot q)} + 2\bar{p} \cdot q\ p \cdot q + \cancel{\bar{p} \cdot p \cdot 2p \cdot q}\right]$$

$$= -4 \quad 2p \cdot q\ 2\bar{p} \cdot q$$

$$\frac{1}{4} \sum_{\text{spin}} |M|^2 = -\frac{1}{4} Q_f^2 e^2 \left(\frac{m_f}{v}\right)^2 \cdot (-4)$$

$$\cdot \left\{ \frac{2\bar{p} \cdot q\ 2p \cdot q}{(2\bar{p} \cdot q)^2} + 2 \cdot \frac{(2\bar{p} \cdot p - 2\bar{p} \cdot q)(2\bar{p} \cdot p - 2p \cdot q)}{2p \cdot q\ 2\bar{p} \cdot q} + \frac{2\bar{p} \cdot q\ 2p \cdot q}{(2p \cdot q)^2} \right\}$$

$$= Q_f^2 e^2 \left(\frac{m_f}{v}\right)^2 \left\{ \frac{2p \cdot q}{2\bar{p}q} + \frac{2\bar{p} \cdot q}{2p \cdot q} + 2 - 2\frac{2\bar{p} \cdot p}{2p \cdot q} - 2\frac{2\bar{p} \cdot p}{2\bar{p}q} + 2\frac{(2\bar{p} \cdot p)^2}{2pq\, 2\bar{p}q} \right\}$$

In term of s, t, u:

$$s = (p+\bar{p})^2 = 2p\bar{p} \qquad t = (\bar{p}-q)^2 = -2\bar{p}\cdot q \qquad u = (p-q)^2 = -2p\cdot q$$

$$\frac{1}{4}\sum|M|^2 = Q_f^2 e^2 \left(\frac{m_f}{v}\right)^2 \left\{ \frac{u}{t} + \frac{t}{u} + 2 + 2\frac{s}{u} + 2\frac{s}{t} + 2\frac{s^2}{ut} \right\}$$

$$= Q_f^2 e^2 \left(\frac{m_f}{v}\right)^2 \left\{ \frac{1}{ut} \right\} \left\{ \underbrace{u^2 + t^2 + 2ut + 2st + 2su}_{\substack{(s+t+u)^2 + s^2 \\ = (m_h^2)^2 + s^2}} + 2s^2 \right\}$$

$$= Q_f^2 e^2 \left(\frac{m_f}{v}\right)^2 \left(\frac{s^2 + m_h^2}{ut} \right)$$

In the CM frame:

$$p = (E, 0\ 0\ E) \qquad k = (\mathcal{E}\ \ q\sin\theta\ \ 0\ \ q\cos\theta)$$

$$\bar{p} = (E, 0\ 0\ -E) \qquad q = (q, -q\sin\theta\ \ 0\ \ -q\cos\theta)$$

$$\mathcal{E} + q = 2E = \sqrt{s} \qquad \mathcal{E}^2 - q^2 = m_h^2 \quad \rightarrow \quad q = \frac{s - m_h^2}{2\sqrt{s}}$$

$$\text{Phase space} = \frac{1}{16\pi}\, d\cos\theta\, \frac{d\phi}{2\pi} \left(\frac{2q}{\sqrt{s}}\right)$$

$$= \frac{1}{16\pi\cdot 2\pi}\, d\cos\theta\, d\phi \left(1 - \frac{m_h^2}{s}\right)$$

$$s = 4E^2 \qquad t = -2Eq(1-\cos\theta) \qquad u = -2Eq(1+\cos\theta)$$

$$ut = 4E^2 q^2 (1-\cos^2\theta)$$

Finally, the color factor: the quark and antiquark must have the same color to annihilate. The probability of this is $\frac{1}{3}$

$$\left.\frac{d\sigma}{d\cos\Theta\, d\phi}\right|_{CM} = \frac{1}{3}\cdot\frac{1}{2s}\cdot\frac{1}{16\pi\cdot 2\pi}\,\frac{2q}{\sqrt{s}}\,Q_f^2 e^2\left(\frac{m_f}{v}\right)^2\frac{s^2+m_h^4}{s\cdot q^2(1-\cos^2\Theta)}$$

$$\left.\frac{d\sigma}{d\cos\Theta\, d\phi}\right|_{cm} = \frac{\alpha}{4\pi s}\,Q_f^2\left(\frac{m_f}{v}\right)^2\left(\frac{2q}{\sqrt{s}}\right)\left(1+\frac{m_h^4}{s^2}\right)\cdot\frac{s}{q^2\sin^2\Theta}$$

with $q = \frac{s-m_h^2}{2\sqrt{s}}$

now $q^2\sin^2\Theta = q_\perp^2$ the γ transverse momentum

we need to convert $d\cos\Theta\, d\phi = \sin\Theta\, d\Theta\, d\phi$ to $d^2 q_\perp$

$$q^1 = q\sin\Theta\cos\phi \qquad q^2 = q\sin\Theta\sin\phi$$

$$\frac{\partial(q^1 q^2)}{\partial(\Theta\,\phi)} = \begin{vmatrix} q\cos\Theta\cos\phi & q\cos\Theta\sin\phi \\ -q\sin\Theta\sin\phi & q\sin\Theta\cos\phi \end{vmatrix} = q^2\cos\Theta\sin\Theta$$

so

$$\frac{d\Theta\sin\Theta\, d\phi}{dq_\perp^1\, dq_\perp^2} = \frac{1}{q^2\cos\Theta} = \frac{1}{q\,(q^2-q_\perp^2)^{1/2}}$$

then for $q + \bar{q} \to h^0 + \gamma$

$$\frac{d^2\sigma}{dq_\perp^2} = \frac{\alpha}{24\pi} Q_f^2 \left(\frac{m_f}{v}\right)^2 \left(1 + \frac{m_h^4}{s^2}\right) \left(\frac{s}{q^2 - q_\perp^2}\right)^{\frac{1}{2}} \frac{1}{q_\perp^2}$$

d.) To estimate this, write

$$\sigma \sim \frac{\alpha}{24\pi} Q_f^2 \left(\frac{m_f}{v}\right)^2 \frac{2\pi}{q_\perp^2} \cdot P_f^2$$

put $Q_f = -\frac{1}{3}$ $m_f = 4.2$ GeV for the b quark

P_f = prob of b in proton ~ 0.1

$q_\perp \sim 100$ GeV

$$\sigma \sim \frac{\alpha}{12} \cdot \frac{1}{9} \cdot \left(\frac{4.2}{246}\right)^2 \cdot 10^{-2} \cdot \frac{1}{(100\text{ GeV})^2}$$

$$\cdot \left[0.197 \text{ GeV} \times 10^{-13}\text{cm}\right]^2$$

$1\text{fm} = 10^{-13}$ cm

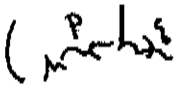
$$\sim 0.76 \times 10^{-15} \times (10^{-26}\text{cm}^2)$$

now 1 barn $= 10^{-24}$ cm^2 so this is

$\sim .0076$ fb

so if the LHC produces enough collisions to give 100 events for $\sigma = 1$ fb we will have 1 event !

e.)

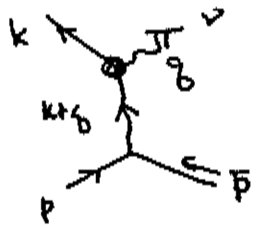
$$p_\mu \left(\right.$$  $$\left. \right) = p_\mu\left(-i\frac{\alpha}{4\pi}C\left(g^{\mu\nu}p\cdot q - p^\nu q^\mu\right)\right)$$

$$= -i\frac{\alpha}{4\pi}\left(p^\nu p\cdot q - p^\nu p\cdot q\right)$$

$$= 0 \quad ✓$$

similarly () $q_\nu = 0$

f.)



$$i\mathcal{M} = (+ie\,Q_f)\cdot\left(-i\frac{\alpha}{4\pi}C\right)\ \bar v(\bar p)\gamma_\mu u(p)\ \frac{-i}{s}$$

$$\cdot\left[g^{\mu\nu}q\cdot(-k-q) - (-k-q)^\nu q^\mu\right]\ \varepsilon^*_\nu(q)$$

$2q\cdot k = (q+k)^2 - m_h^2 = s - m_h^2 \qquad 2p\cdot\bar p = s$

$q^2 = 0$

$q\cdot\varepsilon^*(q) = 0$

$t = -2\bar p\cdot q = m_h^2 - 2p\cdot k$

$u = -2p\cdot q = m_h^2 - 2\bar p\cdot k$

$$= i\frac{e\alpha}{4\pi}Q_f C\ \frac{1}{s}\ \bar v(\bar p)\gamma_\mu u(p)\left[g^{\mu\nu}q\cdot k\ \varepsilon^*_\nu(q) - q^\mu k\cdot\varepsilon^*(q)\right]$$

$$= i\frac{e\alpha}{4\pi}Q_f C\ \frac{1}{s}\ \bar v(\bar p)\gamma_\mu u(p)\left[\varepsilon^{\mu *}(q)\, q\cdot k - q^\mu k\cdot\varepsilon^*(q)\right]$$

$$\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{1}{4}\frac{e^2\alpha^2}{(4\pi)^2} Q_f^2 |C|^2 \frac{1}{s^2} tr[\bar{p\!\!\!/}\gamma_\mu p\!\!\!/ \gamma_\lambda]$$

$$\cdot \sum_{pin}\left[\varepsilon^{\mu *}(q)\, q\cdot k - q^\mu\, k\cdot\varepsilon^*(q)\right]\left[\varepsilon^\lambda(q)\, q\cdot k - q^\lambda\, k\cdot\varepsilon(q)\right]$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \frac{1}{s^2} \cdot 4\left[\bar{p}_\mu p_\lambda + \bar{p}_\lambda p_\mu - g_{\mu\nu}\,\bar{p}\cdot p\right]$$

$$\cdot\left[-g^{\mu\lambda}(q\cdot k)^2 + (q^\mu k^\lambda + q^\lambda k^\mu)\, q\cdot k - q^\mu q^\lambda k^2\right]$$

$$= \frac{1}{4}\frac{\alpha^3}{(4\pi)} Q_f^2 |C|^2 \frac{1}{s^2}$$

$$\cdot\, 4\,\Big\{ (q\cdot k)^2\left[-2\bar{p}\cdot p + 4\bar{p}\cdot p\right]$$

$$+ \quad 2q\cdot k\left[q\cdot\bar{p}\; k\cdot p + q\cdot p\; k\cdot\bar{p} - q\cdot k\; \bar{p}\cdot p\right]$$

$$- \left[2\, q\cdot\bar{p}\; q\cdot p\; k^2 - \underset{0}{q^2}\, k^2\, p\cdot\bar{p}\right]\Big\}$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \frac{1}{s^2} \cdot \Big\{ \cancel{(s-m_h^2)^2\cdot s}$$

$$+ \quad (s-m_h^2)\left[(-t)(m_h^2-t) + (-u)(m_h^2-u) - \cancel{(s-m_h^2)\, s}\right]$$

$$-2m_h^2\, ut \quad \Big\}$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \frac{1}{s^2}\left\{\left[u^2+t^2-(u+t)\, m_h^2\right](s-m_h^2) - 2m_h^2 ut\right\}$$

$$u^2+t^2 = (u+t)^2 - 2ut = (s-m_h^2)^2 - 2ut \qquad u+t = m_h^2 - s$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \frac{1}{s^2}\left\{(s-m_h^2)^3 - 2ut(s\cancel{-m_h^2}) + (s-m_h^2)^2 m_h^2 - \cancel{2ut\, m_h^2}\right\}$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \frac{1}{s^2}\left\{(s-m_h^2)^2\cdot s - 2stu\right\}$$

$$= \frac{1}{4}\frac{\alpha^3}{(4\pi)} Q_f^2 \, |C|^2 \, \frac{1}{s}\left[(s-m_h^2)^2 - 2ut\right]$$

in the CM frame, from p.6

$$(s-m_h^2) = 2\sqrt{s}\, q = 4Eq$$

$$ut = 4E^2q^2(1-\cos\theta)(1+\cos\theta) = 4E^2q^2\sin^2\theta$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 \, |C|^2 \frac{1}{(2E)^2} \, 4E^2q^2\left[4-2\sin^2\theta\right]$$

$$= \frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \cdot 2q^2(2-\sin^2\theta) = \frac{\alpha^3}{2\pi} Q_f^2 |C|^2 q^2 (1+\cos^2\theta)$$

$$\left.\frac{d\sigma}{d\cos\theta\, d\phi}\right|_{cm} = \frac{1}{3}\cdot\frac{1}{2s}\cdot\frac{1}{16\pi\cdot 2\pi}\cdot\frac{2q}{\sqrt{s}}\cdot\frac{\alpha^3}{2\pi} Q_f^2 |C|^2 q^2 (1+\cos^2\theta)$$

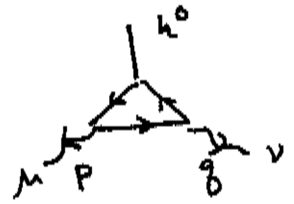
$$= \frac{\alpha^3 Q_f^2 |C|^2}{192\pi^3}\left(\frac{q^3}{s^{3/2}}\right)(1+\cos^2\theta)$$

$$= \frac{\alpha^3 Q_f^2 |C|^2}{1536\pi^3}\left(\frac{2q}{\sqrt{s}}\right)^3(1+\cos^2\theta)$$

$$\frac{d\sigma}{dq_\perp^2} = \frac{\alpha^3 Q_f^2 |C|^2}{1536\pi^3}\left(\frac{2q}{\sqrt{s}}\right)^3 \frac{1}{q(q^2-q_\perp^2)^{1/2}}(1+\cos^2\theta)_{cm}$$

$$= \frac{\alpha^3 Q_f^2 |C|^2}{1536\pi^3}\left(\frac{2q}{\sqrt{s}}\right)^3 \frac{2q^2-q_\perp^2}{q^3(q^2-q_\perp^2)^{1/2}}$$

g.)



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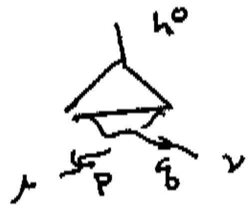
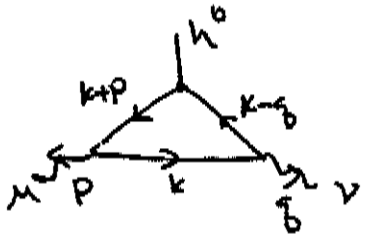




fermion loop   color

$$= \left(+i\,\tfrac{2}{3}e\right)^2 \left(-i\,\frac{m_t}{v}\right) \int \frac{d^4k}{(2\pi)^4} \cdot (-1) \cdot 3$$

$$\cdot\ \mathrm{tr}\left[i\,\frac{(\not k - \not q + m_t)}{(k-q)^2 - m_t^2}\,\gamma^\nu\ \frac{i\,(\not k + m_t)}{k^2 - m_t^2}\,\gamma^\mu\ \frac{i\,(\not k + \not p + m_t)}{(k+p)^2 - m_t^2}\right]$$

$$= -\frac{4}{9}\cdot 3\, e^2\, \frac{m_t}{v} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k-q)^2 - m_t^2]\,[k^2 - m_t^2]\,[(k+p)^2 - m_t^2]}$$

$$\cdot\ \mathrm{tr}\left[(\not k - \not q + m_t)\,\gamma^\nu\,(\not k + m_t)\,\gamma^\mu\,(\not k + \not p + m_t)\right]$$

combine denominators $\qquad x+y+z=1$

$$\text{Denom} = x\left((k-q)^2 - m_t^2\right) + y\left((k+p)^2 - m_t^2\right) + z\left(k^2 - m_t^2\right)$$

$$= k^2 + 2k\cdot(-xq + yp) + xq^2 + yp^2 - m_t^2$$

$$= \mathbb{k}^2 - (-xq + yp)^2 + xq^2 + yp^2 - m_t^2$$

$$= \mathbb{k}^2 + x(1-x)q^2 + y(1-y)p^2 + 2xy\, p\cdot q - m_t^2$$

$$= \mathbb{k}^2 - \Delta \qquad \text{with}$$

$$\Delta = m_t^2 - x(1-x)q^2 - y(1-y)p^2 - 2xy\, p\cdot q$$

$\mathbb{k} = k - xq + yp \qquad$ so $\qquad k = \mathbb{k} + xq - yp$

$$k - q = \mathbb{k} - (1-x)q - yp$$

$$k + p = \mathbb{k} + xq + (1-y)p$$

then

(keep terms 2 + 4's inside the trace.)

$$\mathrm{tr}\,[(\cancel{k}-\cancel{q}+m_t)\,\gamma^\nu(\cancel{k}+m_t)\gamma^\mu(\cancel{k}+\cancel{p}+m_t)]$$

$$= 4\,[\; m_t^3 g^{\mu\nu} + m_t\,(k^\nu (k+p)^\mu + k^\mu (k+p)^\nu - g^{\mu\nu}\,k\cdot(k+p))$$
$$+ m_t\,((k-q)^\nu k^\mu + (k-q)^\mu k^\nu - g^{\mu\nu}\,k\cdot(k-q))$$
$$+ m_t\,((k-q)^\nu (k+p)^\mu - (k-q)^\mu (k+p)^\nu + g^{\mu\nu}(k-q)(k+p))]$$

$$= 4\,[\; m_t^3 g^{\mu\nu} + m_t\,(\mathbb{k}^\mu \mathbb{k}^\nu\cdot 4 - g^{\mu\nu}\,\mathbb{k}^2\cdot 1\,)$$

$$+ m_t\,(\,[xq+(1-y)p]^\mu[xq-yp]^\nu + [xq-yp]^\mu[xq+(1-y)p]^\nu$$
$$-g^{\mu\nu}\,(xq-yp)\cdot(xq+(1-y)p)$$

$$+ [xq-yp]^\mu[-(1-x)q-yp]^\nu + [-(1-x)q-yp]^\mu[xq-yp]^\nu$$
$$-g^{\mu\nu}\,(xq-yp)\cdot(-(1-x)q-yp)$$

$$+ [xq+(1-y)p]^\mu[-(1-x)q-yp]^\nu - [-(1-x)q-yp]^\mu[xq+(1-y)p]^\nu$$
$$+g^{\mu\nu}\,(xq+(1-y)p)\cdot(-(1-x)q-yp))\;]$$

(drop terms linear in $\mathbb{k}$.)

$$= 4\,\Big[\, m_t^3 g^{\mu\nu} + m_t \left(4 k^\mu k^\nu - k^2 g^{\mu\nu}\right)$$

$$+\, m_t \Big(q^\mu q^\nu \cdot \left[x^2 \cdot 2 - x(1-x)\cdot 2 - \cancel{x(1-x)} + \cancel{x(1-x)} \right]$$

$$+\, p^\mu p^\nu \left[-2y(1-y) + 2y^2 - \cancel{y(1-y)} + \cancel{y(1-y)} \right]$$

$$+\, q^\mu p^\nu \left[-xy + x(1-y) - xy + y(1-x) - xy + (1-x)(1-y) \right]$$

$$+\, p^\mu q^\nu \left[x(1-y) - xy + y(1-x) - xy - (1-x)(1-y) + xy \right]$$

$$-\, g^{\mu\nu} q^2 \left[x^2 - \cancel{x(1-x)} + \cancel{x(1-x)} \right]$$

$$-\, g^{\mu\nu} p^2 \left[-\cancel{y(1-y)} + y^2 + \cancel{y(1-y)} \right]$$

$$-\, g^{\mu\nu} p\cdot q \left[x(1-y) - xy - xy + y(1-x) + xy + (1-x)(1-y) \right] \Big) \Big]$$

$$= 4\,\Big[\, m_t^3 g^{\mu\nu} + m_t \left(4 k^\mu k^\nu - k^2 g^{\mu\nu}\right)$$

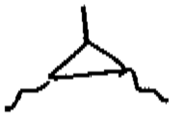
$$+\, m_t \Big(q^\mu q^\nu (4x^2 - 2x) + p^\mu p^\nu (4y^2 - 2y)$$

$$+\, q^\mu p^\nu (1 - 4xy) + p^\mu q^\nu (-1 + 2(x+y) - 4xy)$$

$$-\, g^{\mu\nu} q^2\, x^2 - g^{\mu\nu} p^2\, y^2$$

$$-\, g^{\mu\nu} p\cdot q \left[1 - 2xy \right] \Big) \Big]$$

so

 $$= -\frac{4}{3}\cdot e^2 \frac{m_t}{v}\cdot 4 m_t \int_0^1 dx\,dy\,dz\, \delta(x+y+z-1)$$

$$\cdot \int \frac{d^4k}{(2\pi)^4} \cdot \frac{2}{[k^2-\Delta]^3}$$

$$\Big[(4 k^\mu k^\nu - g^{\mu\nu} k^2) + g^{\mu\nu} m_t^2$$

$$+ q^\mu q^\nu (4x^2-2x) + p^\mu p^\nu (4y^2-2y) + q^\mu p^\nu (1-4xy)$$

$$+ p^\mu q^\nu (-1 + 2(x+y) - 4xy)$$

$$- g^{\mu\nu} q^2 x^2 - g^{\mu\nu} p^2 y^2 - g^{\mu\nu} p\cdot q\, [1-2xy] \Big]$$

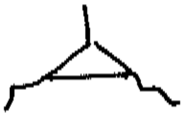
now do the integrals using dimensional regularization:

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2-\Delta]^3} = \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)} \frac{1}{\Delta^{3-\frac{d}{2}}}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^2}{[k^2-\Delta]^3} = \frac{+i}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)} \frac{1}{\Delta^{2-\frac{d}{2}}} \cdot \frac{d}{2}$$

$$\Gamma(3) = 2$$

so that $\rightarrow$



$$= -\frac{16}{3} e^2 \frac{m_t^2}{v} \int dx\, dy\, dz\, \delta(x+y+z-1)$$

$$\cdot \frac{i}{(4\pi)^{d/2}} \cdot \Big\{ \quad \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} \left(4 \cdot \frac{1}{2} g^{\mu\nu} - g^{\mu\nu} \frac{d}{2}\right)$$

$$- \frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-\frac{d}{2}}} m_t^2$$

$$- \frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-\frac{d}{2}}} \left(q^\mu q^\nu (4x^2-2x) + \cdots - g^{\mu\nu} p\cdot q (1-2xy)\right) \Big\}$$

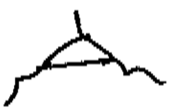
the first term is proportional to

$$\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} \left(2 - \frac{d}{2}\right) g^{\mu\nu} \xrightarrow[d\to 4]{} \frac{1}{\Delta^0} g^{\mu\nu} = \frac{\Delta}{\Delta} g^{\mu\nu}$$

the remaining terms involve

$$\frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-\frac{d}{2}}} \xrightarrow[d\to 4]{} \frac{1}{\Delta}$$

so



$$= -\frac{i}{(4\pi)^2} \frac{16}{3} e^2 \frac{m_t^2}{v} \int dx\, dy\, dz\, \delta(x+y+z-1) \cdot \frac{1}{\Delta}$$

$$\Big\{ \left[\cancel{m_t^2} - x(1-x) q^2 - y(1-y) p^2 - 2xy\, p q\right] g^{\mu\nu} - \cancel{m_t^2} g^{\mu\nu}$$

$$- q^\mu q^\nu (4x^2-2x) - p^\mu p^\nu (4y^2-2y) - q^\mu p^\nu (1-4x) - p^\mu q^\nu (-1+2(x+y)-4xy)$$

$$+ g^{\mu\nu} q^2 x^2 + g^{\mu\nu} p^2 y^2 + g^{\mu\nu} p\cdot q \left[1-2xy\right] \Big\}$$

now we can take the limit $m_t \to \infty$. In this limit

$$\frac{m_t^2}{\Delta} = \frac{m_t^2}{[m_t^2 - x(1-x)q^2 - y(1-y)p^2 - 2xy\,p\cdot q]} \to \frac{m_t^2}{m_t^2} = 1$$

this makes the integrals over x, y, z easy to do

$$\int dx\,dy\,dz\,\delta(x+y+z-1)\;1 = \int_0^1 dx \int_0^{1-x} dy = \int_0^1 dx\,(1-x) = \frac{1}{2}$$

$$\int dx\,dy\,dz\,\delta(x+y+z-1)\;x = \int_0^1 dx\;x\,(1-x) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\int dx\,dy\,dz\,\delta(x+y+z-1)\;x^2 = \int_0^1 dx\;x^2(1-x) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\int dx\,dy\,dz\,\delta(x+y+z-1)\,xy = \int_0^1 dx\;x\int_0^{1-x} dy\;y$$

$$= \int_0^1 dx\;x\,\frac{(1-x)^2}{2} = \int_0^1 dx\;\frac{x}{2}(1-2x+x^2) = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{1}{24}$$

to repeat, let $\langle\,\cdot\,\rangle = \int dx\,dy\,dz\,\delta(x+y+z-1)\;\cdot$

then

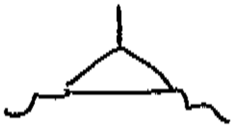
$$\langle 1\rangle = \frac{1}{2}$$

$$\langle x\rangle = \langle y\rangle = \langle z\rangle = \frac{1}{6} \qquad \langle (x+y+z)\rangle = \frac{1}{2}\ \checkmark$$

$$\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle = \frac{1}{12}$$

$$\langle xy\rangle = \langle yz\rangle = \langle xz\rangle = \frac{1}{24}$$

$$\langle (x+y+z)^2\rangle = \langle x^2+y^2+z^2+2xy+2xz+2yz\rangle = 3\cdot\frac{1}{12} + 3\cdot 2\cdot\frac{1}{24} = \frac{1}{2}\ \checkmark$$

 $= -\frac{i}{4\pi}\,\frac{16}{3}\,\alpha\,\frac{1}{\varepsilon}$

$$\cdot\Big\{ g^{\mu\nu}\, q^2 \langle 2x^2 - x\rangle + g^{\mu\nu} p^2 \langle 2y^2 - y\rangle$$

$$+\, g^{\mu\nu}\, p\cdot q \langle 1 - 4xy\rangle$$

$$-\, q^\mu q^\nu \langle 4x^2 - 2x\rangle - p^\mu p^\nu \langle 4y^2 - 2y\rangle - q^\mu p^\nu \langle 1 - 4xy\rangle$$

$$-\, p^\mu q^\nu \langle -1 + 2(x+y) - 4xy\rangle \Big\}$$

now

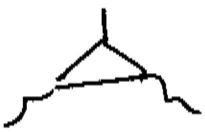
$$\langle 2x^2 - x\rangle = \langle 2y^2 - y\rangle = 2\cdot\frac{1}{12} - \frac{1}{6} = 0$$

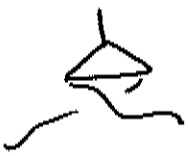
$$\langle 4x^2 - 2x\rangle = \langle 4y^2 - 2y\rangle = 4\cdot\frac{1}{12} - 2\frac{1}{6} = 0$$

$$\langle -1 + 2(x+y) - 4xy\rangle = -\frac{1}{2} + 2\cdot 2\cdot\frac{1}{6} - 4\frac{1}{24} = 0$$

$$\langle 1 - 4xy\rangle = \frac{1}{2} - 4\frac{1}{24} = \frac{1}{3}$$

so!

 $= \frac{-i\alpha}{4\pi}\cdot\frac{16}{3}\cdot\frac{1}{\varepsilon}\cdot\frac{1}{3}\cdot\left(g^{\mu\nu}\, p\cdot q - p^\nu q^\mu\right)$

 = the above, with $\binom{p}{\mu} \leftrightarrow \binom{q}{\nu}$ = the above.

so

 + 

$$= +i\,\frac{\alpha}{4\pi}\left(-\frac{32}{9}\frac{1}{v}\right)\left(g^{\mu\nu}p\cdot q - p^\nu q^\mu\right)$$

so, from the top quark loop, $\quad C = -\frac{32}{9}\frac{1}{v}$

[It turns out that W boson loops give a larger contribution of the opposite sign!]

h.) $\quad i\mathcal{M}(h^0\to\gamma\gamma) =$  $= -i\,\frac{\alpha}{4\pi}\,C\left(g^{\mu\nu}p\cdot q - p^\nu q^\mu\right)$

$\quad\cdot\ \epsilon^*_\mu(p)\,\epsilon^*_\nu(q)$

$$\sum_{\text{spin}}|\mathcal{M}|^2 = \left(\frac{\alpha}{4\pi}\right)^2|C|^2\left(g^{\mu\nu}p\cdot q - p^\nu q^\mu\right)\left(g_{\mu\nu}p\cdot q - p_\nu q_\mu\right)$$

$$= \left(\frac{\alpha}{4\pi}\right)^2|C|^2\cdot\left[4(p\cdot q)^2 - 2(p\cdot q)^2 + \underbrace{p^2q^2}_{=0}\right]$$

$$= \left(\frac{\alpha}{4\pi}\right)^2|C|^2\cdot 2(p\cdot q)^2$$

and $\quad m_h^2 = (p+q)^2 = 2p\cdot q$

$$= \frac{1}{2}\left(\frac{\alpha}{4\pi}\right)^2|C|^2\, m_h^4$$

$$\Gamma(h^0 \to \gamma\gamma) = \frac{1}{2m_h} \cdot \frac{1}{8\pi} \cdot \underbrace{\frac{1}{2}}_{\text{id. particles}} \cdot \frac{1}{2}\left(\frac{\alpha^2}{4\pi}\right) |C|^2\, m_h^4$$

$$= \frac{\alpha^2}{8 \cdot 8 \cdot 16\pi^3} |C|^2\, m_h^3$$

$$= \frac{\alpha^2}{(32)^2 \pi^3} \left(-\frac{32}{9v}\right)^2 m_h^3$$

$$\Gamma(h^0 \to \gamma\gamma) = \frac{\alpha^2}{81\pi^3} \frac{m_h^3}{v^2}$$

numerically:

$$\Gamma(h^0 \to \gamma\gamma) = \underbrace{\frac{m_h}{8\pi}\left(\frac{1\,\text{GeV}}{v}\right)^2}_{79\ \text{keV}} \cdot \underbrace{\frac{8\alpha^2}{81\pi^2}\left(\frac{m_h}{\text{GeV}}\right)^2}_{7.7\times 10^{-3}}$$

$$= 0.61\ \text{keV}$$

$$BR(h^0 \to \gamma\gamma) = (\text{above}) / 4.7\ \text{MeV} = 1.3 \times 10^{-4}$$

[for the real Higgs boson of mass 120 GeV, $BR(h^0 \to \gamma\gamma)$ is enhanced by the W boson loop to about 10^{-3}.]