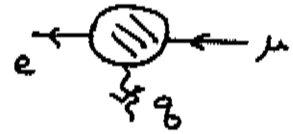
Physics 330 – Final Exam Solutions

a) $$i\mathcal{M} \;=\;$$ 

$$= -ie\,\bar{u}(p')\; i\sigma^{\mu\nu}(-q_\nu)\,\bar{F}_2(q^2)\,u(p)\,\Big|_{q^2=0}$$

$$\sum_{\text{spin}} |\mathcal{M}|^2 = e^2 |\bar{F}_2|^2 \,\mathrm{tr}\{(\not p' + \underset{\text{ignore}}{\not m_e})[-i\sigma^{\mu\nu} q_\nu](\not p + m_\mu)[+i\sigma^{\lambda\sigma} q_\sigma]\}(-g_{\mu\lambda})$$

$$= e^2 |\bar{F}_2(0)|^2\, \mathrm{tr}(\not p' \tfrac{1}{2}[\gamma^\mu, \not q](\not p + m_\mu)\tfrac{1}{2}[\gamma^\lambda, \not q])(-g_{\mu\lambda})$$

the traces with m_μ are 0. The other terms are:

$$= \frac{e^2 |\bar{F}_2(0)|^2}{4}\, \mathrm{tr}[\not p'(\gamma^\mu \not q - \not q \gamma^\mu)\not p(\gamma_\mu \not q - \not q \gamma_\mu)](+1)$$

$$= \frac{e^2 |\bar{F}_2(0)|^2}{4}\, \mathrm{tr}\{\not p' \gamma^\mu \not q \not p \gamma_\mu \not q - \not p' \gamma^\mu \not q \not p \not q \gamma_\mu$$
$$- \not p' \not q \gamma^\mu \not p \gamma_\mu \not q + \not p' \not q \gamma^\mu \not p \not q \gamma_\mu\}$$

$$= \frac{e^2 |\bar{F}_2(0)|^2}{4}\, \mathrm{tr}\{4\not p'\, q\cdot p\, \not q + 2\not p' \not q \not p \not q$$
$$+ 2\not p' \not q \not p \not q + 4\not p' \not q\, p\cdot q\}$$

$$= \frac{e^2\,|\overline{F}(0)|^2}{4} \cdot 4 \cdot \Big\{ 4 \cdot 2 \; p' \cdot q \; p \cdot q$$

$$+ 2 \cdot 2 \cdot \big[2 \cdot p' \cdot q \; p \cdot q - p' \cdot p \underbrace{q^2}_{=0} \big) \Big\}$$

$$= e^2 \, |\overline{F}(0)|^2 \; 16 \; p' \cdot q \; p \cdot q$$

now $\quad m_\mu^2 = p^2 = (p'+q)^2 = 2p' \cdot q = 2(p+q) \cdot q = 2p \cdot q$

so $\quad 2p' \cdot q = 2p \cdot q = m_\mu^2$

$$\sum |M|^2 = 4e^2 \, |\overline{F}_2(0)|^2 \, m_\mu^4$$

$$\Gamma(\mu \to e\gamma) = \frac{1}{2m_\mu} \cdot \underbrace{\frac{1}{8\pi}}_{\text{phase space}} \cdot \frac{1}{2} \sum_{\text{spin}} |M|^2$$

$$= \frac{1}{32\pi \, m_\mu} \cdot 4e^2 \, |\overline{F}_2(0)|^2 \, m_\mu^4$$

$$\Gamma(\mu \to e\gamma) = \frac{\alpha}{2} \, |\overline{F}_2(0)|^2 \, m_\mu^3$$

b.) with $F_2(0) = G \frac{\alpha}{4\pi} \frac{m_\mu}{M^2}$

$$\Gamma(\mu \to e\gamma) = \frac{\alpha^3}{32\pi^2} G^2 \frac{m_\mu^5}{M^4}$$

compare this to

$$\Gamma(\mu) = \frac{\alpha_w^2 m_\mu^5}{384\pi m_w^4} \qquad 384 = 32 \cdot 12$$

$$BR(\mu \to e\gamma) = \frac{12\alpha^3}{\pi\alpha_w^2} G^2 \left(\frac{m_w}{M}\right)^4$$

$$\text{for } \frac{m_w}{M} = \frac{80\text{ GeV}}{200\text{ GeV}} = G^2 \cdot (3.3 \times 10^{-5})$$

$$BR(\tau \to e/\mu\ \gamma) = G^2 \cdot (6.3 \times 10^{-6}) \qquad \left(\begin{matrix}\text{divide by}\\ 5.3\end{matrix}\right)$$

so

$BR(\mu \to e\gamma) < 1.2 \times 10^{-11} \Rightarrow G_{\mu e}^2 < 3.6 \times 10^{-7}$ or $G_{\mu e} < 6 \times 10^{-4}$

$BR(\tau \to e\gamma) < 2.7 \times 10^{-6} \Rightarrow G_{\tau e}^2 < 0.43$ or $G_{\tau e} < 0.66$

$BR(\tau \to \mu\gamma) < 3.2 \times 10^{-7} \Rightarrow G_{\tau\mu}^2 < 5.1 \times 10^{-2}$ or $G_{\tau\mu} < 0.23$

In a theory in which $G_{\mu e} \sim G_{\tau\mu} \sim G_{\tau e}$, the $\mu \to e\gamma$ constraint is obviously the strongest.

Since $(m_\tau/m_\mu)^2 = 280$, the $\mu \to e\gamma$ constraint is stronger even if $G_{\ell \to \mu/e} \sim m_\ell^2$.

c.) As we learned,

$$\sigma(e^+e^- \to \tau^+\tau^-) \underset{s \gg m_\tau^2}{\sim} \frac{4\pi\alpha^2}{3s} = \frac{86.8 \text{ nb}}{[E_{cm}\,(\text{GeV})]^2}$$

$$= 0.78 \text{ nb} \qquad \text{at } E_{cm} = 10.58 \text{ GeV}$$

then 86.3 fb^{-1} gives

$$\# \text{ of events} = (0.78 \times 10^{-9})(86.3 \times 10^{15})$$

$$= 7.0 \times 10^{7} \quad \tau^+\tau^- \text{ pairs}$$

If we could record all of these events and saw no $\tau \to \mu\gamma$ decays. (< 2.3 decays at the 90% CL) we would have a limit of 1.5×10^{-8} (ideal limit)

In practice, the BELLE result is quite an efficient use of the data.

BABAR also has a huge and growing data sample of τ decays — this could be your thesis!

d) The propagators are

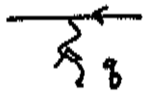
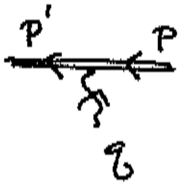
e: $\overleftarrow{\;p\;}$ $\quad \frac{i}{\not{p} - m_e}$ $\qquad \mu$: $\overleftarrow{\;p\;}$ $\quad \frac{i}{\not{p} - m_\mu}$

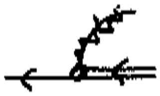
ϕ_e: $\overleftarrow{\;p\;}$ $\quad \frac{i}{p^2 - m^2}$ $\qquad \phi_\mu$: $\overleftarrow{\;p\;}$ $\quad \frac{i}{p^2 - m^2}$

b: $\quad \frac{i}{\not{p} - m_b}$

the vertices are:

photon vertices for $e \quad \mu \quad \phi_e \quad \phi_\mu$:

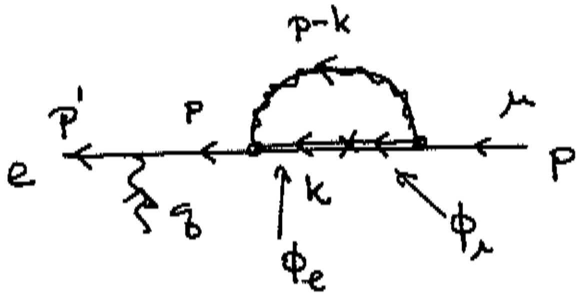
e, μ  $= -ie\gamma^\mu$ $\qquad \phi_e, \phi_\mu$  $= -ie\,(p+p')^\mu$

$e\, b\, \phi_e$  $= ig$  $= ig$

similarly for $\mu b \phi_\mu$

δm^2 : $\quad \phi_\mu$  $\phi_e \quad = -i\,\delta m^2$

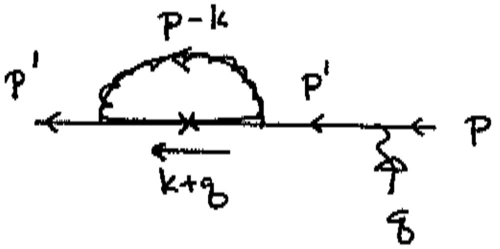
ϕ_e  $\phi_\mu \quad = -i\,\delta m^2$

e.) 

$$= \bar{u}(p')(-ie\gamma^\mu)\,\frac{i(\not{p}+m_e)}{(p^2-m_e^2)}\,(ig)\int\frac{d^4k}{(2\pi)^4}\,\frac{i\,(\not{p}-\not{k}+m_b)}{(p-k)^2-m_b^2}\,ig\,u(p)$$

$$\cdot\ \frac{i}{k^2-m^2}\,(-i\,\delta m^2)\,\frac{i}{k^2-m^2}$$

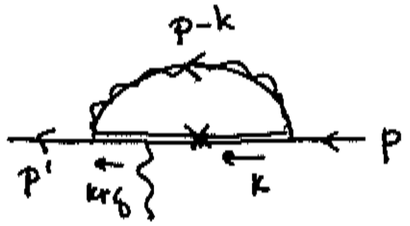
$$= e g^2\,\delta m^2 \int\frac{d^4k}{(2\pi)^4}\,\bar{u}(p')\gamma^\mu\frac{\not{p}+m_e}{p^2-m_e^2}\,\frac{\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2}\,\frac{1}{[k^2-m^2]^2}\,u(p)$$



$$= \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} (ig) \frac{i(\not{p} - \not{k} + m_b)}{(p-k)^2 - m_b^2} (ig) \frac{i(\not{p}' + m_\mu)}{(p')^2 - m_\mu^2} (-ie\gamma^\mu) u(p)$$

$$\cdot \frac{i}{(k+q)^2 - m^2} (-i\delta m^2) \frac{i}{(k+q)^2 - m^2}$$

$$= e g^2 \delta m^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{\not{p} - \not{k} + m_b}{(p-k)^2 - m_b^2} \frac{1}{[(k+q)^2 - m^2]^2} \frac{\not{p}' + m_\mu}{(p')^2 - m_\mu^2} \gamma^\mu u(p)$$



$$= \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} (ig) \frac{i[p - k + m_b]}{(p-k)^2 - m_b^2} ig\, u(p)$$

$$\cdot \frac{i}{(k+q)^2 - m^2} \left[-ie(2k+q)^\mu\right] \frac{i}{k^2 - m^2} (-i\delta m^2) \frac{i}{k^2 - m^2}$$

$$= e g^2 \delta m^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{\not{p} - \not{k} + m_b}{(p-k)^2 - m_b^2} \frac{(2k+q)^\mu}{(k+q)^2 - m^2 \, (k^2 - m^2)^2} u(p)$$

$$= \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} (ig) \frac{i\, \not{p}-\not{k}+m_b}{(p-k)^2-m_b^2} (ig)\, u(p)$$

$$\cdot \frac{i}{(k+q)^2-m^2} (-i\delta m^2) \frac{i}{(k+q)^2-m^2} \; -ie(2k+q)^\mu \frac{i}{k^2-m^2}$$

$$= eg^2 \delta m^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2} \frac{(2k+q)^\mu}{[(k+q)^2-m^2]^2(k^2-m^2)} u(p)$$

f.) In the first diagram, we: $\qquad q = p'-p$

$$q_\mu \cdot \quad \gamma^\mu \frac{\not{p}+m_e}{p^2-m_e^2} = (\not{p}'-\not{p}) \frac{\not{p}+m_e}{p^2-m_e^2}$$

$$= [(\not{p}'-m_e) - (\not{p}-m_e)] \frac{1}{\not{p}-m_e}$$

now $\quad \bar{u}(p')(\not{p}'-m_e) = 0$

so

$q_\mu \cdot$  $= -eg^2\delta m^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2} \frac{1}{(k^2-m^2)^2} u(p)$

In the second diagram in (e), use

$$\frac{(\not p'+m_\mu)}{(p')^2-m_\mu^2}\,\gamma^\mu q_\mu\, u(p)$$

$$= \frac{1}{\not p'-m_\mu}\left[(\not p'-m_\mu)-(\not p-m_\mu)\right]u(p) \quad = \quad u(p)$$

using $\quad (\not p-m_\mu)\,u(p)=0$

then

$q_\mu \cdot$  $= + e g^2 \delta m^2 \displaystyle\int\frac{d^4k}{(2\pi)^4}\,\bar u(p')\frac{p-k+m_b}{(p-k)^2-m_b^2}\,\frac{1}{\left[(k+q)^2-m^2\right]^2}\,u(p)$

In the last two diagrams, use:

$$q^\mu\,(2k+q)_\mu = 2k\cdot q+q^2 = (k+q)^2-k^2$$
$$= \left[(k+q)^2-m^2\right]-\left[k^2-m^2\right]$$

so

$q_\mu\,\Big($  $+$  $\Big)$

$$= e g^2 \delta m^2 \int\frac{d^4k}{(2\pi)^4}\,\bar u(p')\frac{\not p-\not k+m_b}{(p-k)^2-m_b^2}\,\left[((k+q)^2-m^2)-(k^2-m^2)\right]\,u(p)$$

$$\left(\frac{1}{\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]^2}+\frac{1}{\left[(k+q)^2-m^2\right]^2(k^2-m^2)}\right)$$

$$= e g^2 \delta m^2 \int\frac{d^4k}{(2\pi)^4}\,\bar u(p')\,\frac{p-k+m_b}{(p-k)^2-m_b^2}\,u(p)$$

$$\left\{\frac{1}{\left[k^2-m^2\right]^2}-\frac{1}{\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]}+\frac{1}{\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]}-\frac{1}{\left[(k+q)^2-m^2\right]^2}\right\}$$

or

q_μ ( + )

$$= e g^2 \delta m^2 \int \frac{d^4k}{(2\pi)^4}\, \bar{u}(p')\, \frac{\not p - \not k + m_b}{(p-k)^2 - m_b^2} \left(\frac{1}{[k^2-m^2]^2} - \frac{1}{(k+q)^2 - m^2} \right) u(p)$$

which cancels the contribution from the first two diagrams.

g.) The Gordon identity is: $(\not p\, u(p) = m_\mu u(p) \quad \bar{u}(p')\not p' = \bar{u}(p') m_e)$

$$\bar{u}(p')[\not p' \gamma^\mu + \gamma^\mu \not p]\, u(p) = (m_e + m_\mu)\, \bar{u}(p')\gamma^\mu u(p)$$

$$= \bar{u}(p')\left(\tfrac{1}{2}\{\not p', \gamma^\mu\} + \tfrac{1}{2}\{\gamma^\mu, \not p\} - \tfrac{1}{2}[\gamma^\mu, \not p' - \not p] \right) u(p)$$

$$= \bar{u}(p')\left((p+p')^\mu + i\sigma^{\mu\nu} q_\nu \right) u(p)$$

later we'll use this in the form

$$\bar{u}(p')\, (p+p')^\mu\, u(p) = (m_e + m_\mu)\, \bar{u}(p')\gamma^\mu u(p) - \bar{u}(p')\, i\sigma^{\mu\nu} q_\nu\, u(p)$$

this will allow us to put the sum of diagrams into the form:

$$-ie\, \bar{u}(p')[\gamma^\mu F_1(q^2) + q^\mu F_3(q^2) + i\sigma^{\mu\nu} q_\nu F_2(q^2)]\, u(p)$$

$$q^\mu(\text{this}) = 0 \Rightarrow \bar{u}(p')[\not q F_1(q^2) + q^2 F_3(q^2) + 0]\, u(p) = 0$$

$$\bar{u}(p')[(\not p' - \not p) F_1(q^2) + q^2 F_3(q^2)]\, u(p) = 0$$

$$(m_e - m_\mu) F_1(q^2) + q^2 F_3(q^2) = 0$$

or

$$F_1(q^2) = \frac{q^2}{m_\mu - m_e} F_3(q^2)$$

if $F_3(q^2)$ has a smooth limit as $q^2 \to 0$, $F_1(q^2) \propto q^2 \to 0$ as $q^2 \to 0$.

h.) Now we have to get to work computing diagrams. A little table of integrals will be useful:

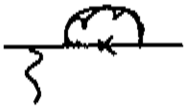
$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - \Delta]^3} = \frac{-i}{(4\pi)^2} \cdot \frac{1}{2} \frac{1}{\Delta}$$

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - \Delta]^4} = \frac{i}{(4\pi)^2} \cdot \frac{1}{3 \cdot 2} \frac{1}{\Delta^2}$$

$$\int \frac{d^4 k}{(2\pi)^4} \frac{k^2}{[k^2 - \Delta]^4} = \frac{-i}{(4\pi)^2} \cdot \frac{2}{3 \cdot 2} \frac{1}{\Delta}$$

$$\int \frac{d^4 k}{(2\pi)^4} \frac{k^\mu k^\nu}{[k^2 - \Delta]^4} = \frac{-i}{(4\pi)^2} \frac{1}{3 \cdot 2} \frac{1}{\Delta} \cdot \frac{1}{2} g^{\mu\nu}$$

then

 $$= e g^2 \, \delta m^2 \, \bar{u}(p') \gamma^\mu \frac{\not p + m_e}{p^2 - m_e^2} \int \frac{d^4k}{(2\pi)^4} \frac{\not p - \not k + m_b}{[(p-k)^2 - m_b^2](k^2-m^2)^2} u(p)$$

combine denominators using z:

$$\frac{1}{[(p-k)^2 - m_b^2][k^2-m^2]^2} = \int_0^1 dz \, \frac{2(1-z)}{[z((p-k)^2-m_b^2) + (1-z)k^2 - m^2]^3}$$

the denominator is: $$z(k-p)^2 - z m_b^2 + (1-z)k^2 - (1-z)m^2$$

$$= k^2 - 2k\cdot zp + zp^2 - (z m_b^2 + (1-z)m^2)$$

$$= \mathbb{k}^2 + \underbrace{z(1-z)p^2 - (z m_b^2 + (1-z)m^2)}_{-\Delta}$$

$$\mathbb{k} = k - zp$$
$$k = \mathbb{k} + zp$$
$$k - p = \mathbb{k} - (1-z)p$$

integrates to 0

so

$$= e g^2 \, \delta m^2 \, \bar{u}(p') \gamma^\mu \frac{\not p + m_e}{p^2 - m_e^2} \int_0^1 dz \, 2(1-z) \int \frac{d^4\mathbb{k}}{(2\pi)^4} \frac{((1-z)\not p - \not{\mathbb{k}} + m_b)}{[\mathbb{k}^2 - \Delta]^3} u(p)$$

$$= \frac{-ieg^2}{(4\pi)^2} \delta m^2 \, \bar{u}(p') \gamma^\mu \frac{\not p + m_e}{p^2 - m_e^2} \int_0^1 dz \, \frac{(1-z)\,[(1-z)\not p + m_b]}{[z m_b^2 + (1-z)m^2 - z(1-z)p^2]} u(p)$$

now use $\quad \not p \, u(p) = m_\mu \, u(p) \qquad p^2 = m_\mu^2$

$$\frac{\not p + m_e}{p^2 - m_e^2} = \frac{1}{m_\mu - m_e}$$

so

 $= (-ie)\ \dfrac{g^2\,\delta m^2}{(4\pi)^2}\ \dfrac{1}{m_\mu - m_e}\ \bar{u}\,\gamma^\mu u$

$$\cdot \int_0^1 dz\ \frac{(1-z)\ \left((1-z)m_\mu + m_b\right)}{\left[z m_b^2 + (1-z)\, m^2 - z(1-z)\, m_\mu^2\right]}$$

 $= e\, g^2\, \delta m^2 \displaystyle\int \frac{d^4k}{(2\pi)^4}\ \bar{u}(p')\, \frac{(\not{p}' - \not{k} + m_b)}{(p-k)^2 - m_b^2}\, \frac{1}{\left[(k+q)^2 - m^2\right]^2}$

$$\cdot \frac{\not{p}' + m_\mu}{p'^2 - m_\mu^2}\,\gamma^\mu\, u(p)$$

combine denominators with:

$$\frac{1}{\left[(p-k)^2 - m_b^2\right]\left[(k+q)^2 - m^2\right]^2} = \int_0^1 dz\ \frac{2(1-z)}{\left[z\left((p-k)^2 - m_b^2\right) + (1-z)\left((k+q)^2 - m^2\right)\right]^3}$$

the denominator is: $k^2 + 2k\cdot(-zp + (1-z)q) + z p^2 + (1-z)q^2 - (z m_b^2 + (1-z)m^2)$

$$= \mathbb{k}^2 - (zp - (1-z)q)^2 + zp^2 + (1-z)\, q^2 - (z m_b^2 + (1-z)\, m^2)$$

$$= \mathbb{k}^2 + (z - z^2)\, p^2 + 2p\cdot q\, z(1-z) + (1-z) z\, q^2 - (z m_b^2 + (1-z) m^2)$$

$$= \mathbb{k}^2 + z(1-z)(p+q)^2 - z m_b^2 + (1-z) m^2$$

$$\mathbb{k} = k - zp + (1-z)q$$

$$k = \mathbb{k} + zp - (1-z)q$$

$$k - p = \mathbb{k} - (1-z)(p+q) = \mathbb{k} - (1-z)\,p'$$

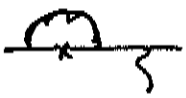
So

$$= e g^2 \delta m^2 \int_0^1 dz\; 2(1-z) \int \frac{d^4 k}{(2\pi)^4} \frac{\bar u\,[\not p'(1-z) - \not k + m_b]}{[k^2 - D]^3}$$

$$\cdot \frac{\not p' + m_\mu}{p'^2 - m_\mu^2}\; u(p)$$

again the $\not k$ term integrates away. Then we can set $\bar u \not p' = \bar u\, m_e$

and $\frac{\not p' + m_\mu}{p'^2 - m_\mu^2} = \frac{1}{m_e - m_\mu}$



$$= (-ie)\,\frac{g^2 \delta m^2}{(4\pi)^2}\left(\frac{-1}{m_\mu - m_e}\right)\bar u \gamma^\mu u$$

$$\cdot \int_0^1 dz\; (1-z)\, \frac{((1-z)m_e + m_b)}{[z m_b^2 + (1-z)m^2 - z(1-z)m_e^2]}$$

For the remaining two integrals, we need to combine denominators according to:

$$\frac{1}{[(p-k)^2 - m_b^2]\,[(k+q)^2 - m^2]\,[k^2 - m^2]} = \int dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{[z((p-k)^2 - m_b) + x((k+q)^2 - m^2) + y(k^2 - m^2)]^3}$$

$$\frac{1}{((p-k)^2 - m_b^2)((k+q)^2 - m^2)(k^2 - m^2)^2} = \int dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2\cdot 3y}{[\quad]^4}$$

$$\frac{1}{((p-k)^2 - m_b^2)\,((k+q)^2 - m^2)^2\,(k^2 - m^2)} = \int dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2\cdot 3x}{[\quad]^4}$$

the denominator in all cases is:

$$z((p-k)^2-m_b^2) + x((k+q)^2-m^2) + y(k^2-m^2)$$

$$= k^2 + 2k(-zp+xq) + zp^2 + xq^2 - [zm_b^2 + (x+y)m^2]$$

$$= \ell^2 - (zp-xq)^2 + zp^2 + xq^2 - [zm_b^2 + (1-z)m^2]$$

$$= \ell^2 + z\underbrace{(1-z)}_{x+y}p^2 + 2p\cdot q\,xz + x\underbrace{(1-x)}_{y+z}q^2 - [zm_b^2 + (1-z)m^2]$$

$$= \ell^2 + yzp^2 + xz(p+q)^2 + xyq^2 - (zm_b^2 + (1-z)m^2)$$

$$= \ell^2 - \{zm_b^2 + (1-z)m^2 - xyq^2 - xzm_e^2 - yzm_\mu^2\}$$

$$\ell = k - zp + xq$$

$$k = \ell + zp - xq$$

$$k-p = \ell - \underbrace{(1-z)}_{x+y}p - xq = \ell - xp' - yp$$

$$2k+q = 2\ell + 2zp + \underbrace{(1-2x)}_{y+z-x}q = 2\ell + z(p+p') + (y-x)q$$

then

 +  $= eg^2\,\delta m^2 \int dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}$

$$\cdot\, 6(x+y) \quad \cdot\, \bar{u}(p')\,\frac{(x\not{p}' + y\not{p} - \not{\ell} + m_b)[2\ell + z(p+p') + (y-x)q]^\mu}{[\ell^2-\Delta]^4}$$

terms linear in ℓ integrate to 0.

$$= (-ie)\,\frac{g^2 \delta m^2}{(4\pi)^2} \int_0^1 dx\,dy\,dz\;\delta(x+y+z-1)\cdot \underbrace{(x+y)}_{=(1-z)}$$

$$\Big\{\; \bar{u}(p')\,(-\gamma^\mu)\left(\frac{2}{2} g^{\mu\nu}\right) u(p)\;\frac{1}{\Delta}$$

$$-\frac{1}{\Delta^2}\,\bar{u}(p')\left[x m_e + y m_\mu + m_b\right] u(p)\;\left[z(p'+p)^\mu + (y-x)\gamma^\mu\right]\Big\}$$

$$= (-ie)\,\frac{g^2 \delta m^2}{(4\pi)^2} \int dx\,dy\,dz\;\delta(x+y+z-1)$$

$$\Big\{\; \bar{u}\gamma^\mu u \cdot \left(-\frac{1}{\Delta}\right)(1-z)$$

$$+\;\bar{u}\gamma^\mu u \cdot (x-y)\left(x m_e + y m_\mu + m_b\right)\frac{1}{\Delta^2}(1-z)$$

$$+\;\bar{u}(p+p')^\mu u \cdot (-z)\left(x m_e + y m_\mu + m_b\right)\frac{(1-z)}{\Delta^2}\Big\}$$

$$= (-ie)\,\frac{g^2 \delta m^2}{(4\pi)^2} \int dx\,dy\,dz\;\delta(x+y+z-1)\;(1-z)$$

$$\cdot\Big\{\; \bar{u}\gamma^\mu u\left(-\frac{1}{\Delta} + \frac{(m_e+m_\mu)(-z)(x m_e + y m_\mu + m_b)}{\Delta^2}\right)$$

$$+\;\bar{u}\gamma^\mu u\;\frac{(x-y)\,(x m_e + y m_\mu + m_b)}{\Delta^2}$$

$$+\;\bar{u}\,i\sigma^{\mu\nu}q_\nu u\;\frac{z\,(x m_e + y m_\mu + m_b)}{\Delta^2}\Big\}$$

$$\Delta = z m_b^2 + (1-z) m^2 - xy\,q^2 - xz\,m_e^2 - yz\,m_\mu^2$$

Now combine the three contributions:

$$= (-ie)\ \bar{u}\left[\gamma^\mu F_1 + q^\mu F_3 + i\sigma^{\mu\nu} q_\nu F_2\right] u$$

where

$$F_1 = \frac{g^2\,\delta m^2}{(4\pi)^2}\left\{ \int dz\, \frac{(1-z)}{m_\mu - m_e}\left(\frac{(1-z)m_\mu + m_b}{\left[z m_b^2 + (1-z)m^2 - z(1-z)m_\mu^2\right]} - \frac{(1-z)m_e + m_b}{\left[z m_b^2 + (1-z)m^2 - z(1-z)m_e^2\right]} \right) \right.$$

$$\left. - \int dx\,dy\,dz\ \delta(x+y+z-1)\left(\frac{(1-z)}{\Delta} + \frac{z(1-z)(m_\mu + m_e)(x m_e + y m_\mu + m_b)}{\Delta^2} \right) \right\}$$

$$F_3 = \frac{g^2\,\delta m^2}{(4\pi)^2}\int dx\,dy\,dz\ \delta(x+y+z-1)\ \frac{(x-y)(x m_e + y m_\mu + m_b)(1-z)}{\Delta^2}$$

$$F_2 = \frac{g^2\,\delta m^2}{(4\pi)^2}\int dx\,dy\,dz\ \delta(x+y+z-1)\ \frac{z(1-z)(x m_e + y m_\mu + m_b)}{\Delta^2}$$

with $\Delta = z m_b^2 + (1-z)m^2 - xy\,q^2 - xz\,m_e^2 - yz\,m_\mu^2$

since Δ is positive in the region we are considering of

$m_b^2, m^2 \gg m_\mu^2, m_e^2$

these expressions are nonsingular as $q^2 \to 0$

i.) Now compute $F_1(0)$ in the limit $m_e \to 0$ then $m_\mu \to 0$

$$F_1(0) = \frac{g^2 \delta m^2}{(4\pi)^2} \Big\{ \int_0^1 dz \; \frac{(1-z)}{m_\mu} \Big[\frac{(1-z)\, m_\mu}{[z m_b^2 + (1-z) m^2]} + \mathcal{O}(m_\mu^2) \Big]$$

$$- \int_0^1 dz \int_0^{1-z} dx \; \frac{1}{[z m_b^2 + (1-z) m^2]}$$

$$- \mathcal{O}(m_\mu) \Big\}$$

$$= \frac{g^2 \delta m^2}{(4\pi)^2} \int_0^1 dz \Big\{ \frac{(1-z)^2}{[z m_b^2 + (1-z) m^2]} - \frac{(1-z)\cdot(1-z)}{[z m_b^2 + (1-z) m^2]} \Big\}$$

$$= 0$$

j.) (See below)

k.) For $\mu \to e\gamma$, the photon has $q^2 = 0$. But $F_1(q^2) = 0$ at $q^2 = 0$. F_3 multiplies q^μ, and $q^\mu \cdot \varepsilon^*(q) = 0$. So neither term contributes to $\mu \to e\gamma$.

l.) In the limit $m_e \to 0$ $m_\nu \to 0$ $q^2 \to 0$

$$\bar{F}_2(0) = \frac{g^2 \delta m^2}{(4\pi)^2} \int_0^1 dz \int_0^{1-z} dx \frac{z(1-z)\, m_b}{[z m_b^2 + (1-z) m^2]^2}$$

$$= \frac{g^2 \delta m^2}{(4\pi)^2} \int_0^1 dz \frac{z(1-z)^2 \, m_b}{[z m_b^2 + (1-z) m^2]^2}$$

Now this is a simple integral:

$$\mathrm{II} = \int_0^1 dz \frac{z(1-z)^2}{[z m_b^2 + (1-z) m^2]^2} = \frac{m^4 + 4m^2 m_b^2 - 5m_b^4 + (2m_b^4 + 4m^2 m_b^2)\log\frac{m_b^2}{m^2}}{2(m^2 - m_b^2)^4}$$

so

$$\bar{F}_2(0) = \frac{1}{4\pi} \cdot \frac{g^2}{(4\pi)} \cdot m_b \cdot \left(\frac{\delta m^2}{m^4}\right)$$

$$\cdot \frac{1}{2(1 - m_b^2/m^2)^4} \left(1 + 4\frac{m_b^2}{m^2} - 5\frac{m_b^4}{m^4} + 2\left(\frac{m_b^4}{m^4} + 2\frac{m_b^2}{m^2}\right) \log\frac{m_b^2}{m^2}\right)$$

which is of the form

$$\bar{F}_2(0) = G \frac{\alpha}{4\pi} \cdot m_b \frac{1}{m^2}$$

with $G \sim \left(\frac{\delta m^2}{m^2}\right)$

now there is one unfinished piece of business:

the extra credit part (j)

We need to simplify the expression for $F_1(q^2)$ on p. 16

First, let $\quad x = (1-z)\xi \qquad y = (1-z)(1-\xi)$

$$\int dx\,dy\,dz\,\delta(x+y+z-1) = \int_0^1 dz \int_0^{1-z} dx = \int_0^1 dz \int_0^1 d\xi\,(1-z)$$

now take the term

$$\int dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{(1-z)}{\Delta}$$

$$= \int_0^1 dz \int_0^1 d\xi\,\frac{(1-z)^2}{\left[z m_b^2 + (1-z)m^2 - z(1-z)(\xi m_e^2 + (1-\xi)m_\mu^2) - (1-z)^2\xi(1-\xi)q^2\right]}$$

and integrate by parts in ξ:

$$= \int_0^1 dz \left\{ \left.\frac{(1-z)\left[(1-z)(\xi m_e + (1-\xi)m_\mu) + m_b\right]/(m_e - m_\mu)}{\left[z m_b^2 + (1-z)m^2 - z(1-z)(\xi m_e^2 + (1-\xi)m_\mu^2) - (1-z)^2\xi(1-\xi)q^2\right]}\right|_{\xi=0}^{\xi=1} \right.$$

$$\left. - \int_0^1 d\xi\,\frac{(1-z)}{m_e - m_\mu}\,\frac{\left[(1-z)(\xi m_e + (1-\xi)m_\mu) + m_b\right]\left[z(1-z)(m_e^2 - m_\mu^2) + (1-z)^2(1-2\xi)q^2\right]}{\left[z m_b^2 + (1-z)m^2 - z(1-z)(\xi m_e^2 + (1-\xi)m_\mu^2) - (1-z)^2\xi(1-\xi)q^2\right]^2} \right\}$$

this gives a nice identity:

$$\int_0^1 dz \int_0^{1-z} dx \quad \frac{(1-z)}{\Delta}$$

$$= \int_0^1 dz \quad \frac{(1-z)}{(m_\mu - m_e)} \left\{ \frac{(1-z)m_\mu + m_b}{[z m_b^2 + (1-z) m^2 - z(1-z) m_\mu^2]} - \frac{((1-z) m_e + m_b)}{[z m_b^2 + (1-z) m^2 - z(1-z) m_e^2]} \right\}$$

cancels the first term of F_1

$$- \int_0^1 dz \int_0^{1-z} dx \quad (1-z) \left\{ \frac{(x m_e + y m_\mu + m_b)}{\Delta^2} \right\}$$

$$\cdot \left\{ \underbrace{\frac{1}{m_\mu - m_e} \left[z(1-z)(m_\mu^2 - m_e^2) - (1-z)(y-x)\, q^2 \right]} \right\}$$

$$= z(1-z)(m_\mu + m_e)$$

so!

$$\frac{F_1(q^2)}{[g^2 8\pi^2 (4\pi)^2]} = \int_0^1 dz \int_0^{1-z} dx \quad (1-z) \quad \frac{q^2}{(m_\mu - m_e)} \quad \frac{(x-y)(x m_e + y m_\mu + m_b)}{\Delta^2}$$

$$= \frac{q^2}{m_\mu - m_e} \int dx\, dy\, dz\, \delta(x+y+z-1) \ \frac{(x-y)(x m_e + y m_\mu + m_b)}{\Delta^2}$$

or $$F_1(q^2) = \frac{q^2}{m_\mu - m_e} \ F_3(q^2)$$ Q.E.D.